MATHEMATICS MAGAZINE

Index to Volume 24   September 1950 - June 1951

Articles by Title

Articles - by Authors

The digit after the Author's name indicates the number of this volume in which his biographical sketch appears. When the digit is missing, the sketch was published in a previous volume.

Current Papers and Books

PROBLEMS AND QUESTIONS

Edited by C. W. Trigg

Contributors

Numbers refer to pages. Bold type indicates a problem solved and solution published; italics, a problem solved but solution not published; ordinary type, a problem proposed.

Solutions

The page on which a solution appears is in the partntheses following the number of the problem.

Quickies

VOL. XXIV, No. 5, May–June, 1951

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to some member of the Executive Committee.

Manuscripts should be typed on 8½″ x 11″ paper, double-spaced with 1″ margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are $3.00, sponsoring subscriptions $10.00, single copies 65¢. Reprints: 25 free to authors; 100 @ $1.25 per page with a minimum of $6.00. All reprints are bound.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, Calif.

Advertisers should contact the managing editor.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

CONTENTS

THE DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CALIFORNIA - LOS ANGELES

announces an

INSTITUTE FOR TEACHERS OF MATHEMATICS

July 9 - 20, 1951

on the

LOS ANGELES CAMPUS - UNIVERSITY OF CALIFORNIA

DIRECTOR: W.W. Rankin, Professor of Mathematics, Duke University.

Two lectures daily will be given by outstanding men from business, industry, and the teaching profession. Study groups will be conducted daily by leaders of recognized ability chosen from teachers in the various fields of mathematical instruction.

Certificates of attendance will be issued to those registered in the Institute. It will also be possible for registrants to earn two units of college credit by attending lectures and by participating in a sufficient number of the study groups.

Application blanks for the Institute and further details may be obtained by writing to

Clifford Bell
Mathematics Department
University of California
Los Angeles, 24, California

****

THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN NEW ENGLAND

will sponsor

THE INSTITUTE FOR TEACHERS OF MATHEMATICS

August 23 - 30

at CONNECTICUT COLLEGE, NEW LONDON, CONN.

If you wish a program, please drop a card to

Charles H. Mergendahl
Newton High School
Newtonville, 60, Mass.

or

Katherine E. O'Brien
Deering High School
Portland, Maine

OUR CONTRIBUTORS -

(Cont. from inside back cover)

Richard F. Arens, Assistant Professor of Mathematics, University of California, Los Angeles, was born in Iserlohn, Germany in 1919. A graduate of University of California, Los Angeles (A.B. '41) and Harvard (A.M. '42, Ph.D. '45), Dr. Arens was a teaching fellow and tutor at Harvard, 1942-45 and then served as a staff assistant at the Institute for Advanced Study before being appointed to his present position. His special mathematical interests include topology and topological algebra, functional analysis, and harmonic analysis on groups.

Harriet F. Montague, Professor of mathematics, has been a member of the faculty of the University of Buffalo since receiving her B.S. and M.A. degrees there. The only interruption in this service came during the time taken to study for and receive the Ph.D. degree from Cornell University in 1935. Although her field of concentration is geometry, particularly algebraic geometry, Professor Montague has devoted considerable time to the development of the tutorial system in mathematics at the University of Buffalo. Her paper, which appeared in the March-April issue, is an explanation of that system.

Irving S. Reed was educated at the California Institute of Technology (B.S. '44; Ph.D. '49). After working as a mathematician with Northrop Aircraft and teaching mathematics part time at the University of Southern California, Dr. Reed became Chief Mathematician and Director of the Computer Research Corporation in 1950. He is a member of Sigma Xi, the American Mathematical Society and the Association for Computing Machinery. His article on groups appeared in the March-April issue.

SOLUTION OF A FUNCTIONAL EQUATION IN THE MULTIPLICATIVE THEORY OF NUMBERS

E. T. Bell

1. To recall some definitions: $f(x)$ is a *numerical function* of x if $f(x)$ is single-valued and finite for positive integer values n of x. If the integers m, n, have the greatest common divisor 1, written $(m, n) = 1$, and if $f(mn) = f(m)f(n)$, $f(x)$ is said to be *factorable*,

$$f(mn) = f(m)f(n), \qquad (m, n) = 1.$$

Further, if $f(1) \neq 0$, $f(x)$ is called *regular*, and without loss of generality, $f(1)$ may be taken equal to 1. This note is concerned with numerical functions all of which, unless otherwise noted, are restricted to be factorable and regular. By convention if, as for Euler's $\phi(n)$, a function is not otherwise defined for $n = 1$, the value is 1; thus $\phi(1) = 1$, $\mu(1) = 1$, where μ is the Möbius function. Also $0^0 = 1$ by definition.

Several of the numerical functions occurring most frequently in the literature[1] are factorable and regular. For example, if

$$n = \Pi_i p_i^{s_i}, \tag{1.1}$$

is the prime-factor decomposition of n, there are the wellknown functions $\mu(n)$, $= 0$ if n is divisible by a square greater than 1, and otherwise $= 1$ or -1, according as the number of prime divisors of n is even or odd; $\eta(n)$, $= 1$ or 0 according as $n = 1$ or $n > 1$; $u_r(n)$, $= n^r$; $\lambda(n)$ (Liouville's function), $= 1$ or -1 according as the total number of prime divisors of n is even or odd; $\phi(n)$ (Euler's function), the number of integers not exceeding n and prime to n. There are many more.

2. There is an extensive algebra of these functions, in which two species of multiplication dominate the development.

The *D*, or *Dirichlet*, *product* of the numerical functions $F(n)$, $G(n)$, whether factorable or not, is the numerical function $H(n)$,

$$H(n) \equiv \sum_{\alpha\beta=n} F(\alpha)G(\beta), \qquad n = 1, 2, \cdots, \tag{2.1}$$

the sum referring to all decompositions of n into a pair of positive integer divisors α, β. The entire theory of this kind of multiplication originated in Gauss' theorem (as it may be written) that

$$\sum_{\alpha\beta=n} u_0(\alpha)\phi(\beta) = u_1(n).$$

If $R(n)$, $S(n)$ are any numerical functions, the equality $R = S$ means

$$R(n) = S(n), \qquad n = 1, 2, \cdots,$$

1. See L. E. Dickson, *History of the Theory of Numbers*, Vol. 1, 1919, Chapters 5, 9, 19.

We write (2.1) as $H = FG$, thus defining the D product FG of F, G. This multiplication, as easily shown, is associative and commutative.

The second kind of multiplication gives the *absolute product*, $|FG|$, defined by

$$|FG|(n) = F(n)G(n), \qquad n = 1, 2, \cdots$$

3. Returning to the first section, we remark that all the functions noted there, and ∞^2 more, are special cases of the following $F(n)$, where $a(x)$, $b(x)$ are any functions which are single valued and finite for positive prime values of x,

$$F(n) \equiv \prod_i [b(p_i)]^{s_i - 1} [b(p_i) - a(p_i)], \tag{3.1}$$

the notation being as in (1.1). For example, as immediately verified,

$$\begin{array}{lll} a(p) = 1, & b(p) = 0, & F(n) = \mu(n); \\ a(p) = -1, & b(p) = 0, & F(n) = [\mu(n)]^2; \\ a(p) = 0, & b(p) = 1, & F(n) = u_0(n); \\ a(p) = 0, & b(p) = 0, & F(n) = \eta(n), \end{array}$$

$= 1$ if $n = 1$, $= 0$ if $n > 1$;

$$\begin{array}{lll} a(p) = 0, & b(p) = p^r, & F(n) = u_r(n); \\ a(p) = 0, & b(p) = 1, & F(n) = \lambda(n); \\ a(p) = 1, & b(p) = p, & F(n) = \phi(n). \end{array}$$

For reasons that need not be gone into here, $F(x)$ is said to be of *linear type*. Let $G(n)$ also be of linear type,

$$G(n) \equiv \prod_i [c(p_i)]^{s_i - 1} [d(p_i) - c(p_i)].$$

4. The functional equation which is the subject of this note is, with F, G as above,

$$FG = |FG|, \tag{4.1}$$

or, with the notation of section 2,

$$\sum_{\alpha\beta = n} F(\alpha)G(\beta) = F(n)G(n), \quad n = 1, 2, \cdots,$$

in which $F(x)$, $G(x)$ are both of linear type.

By an immediate extension, we might consider the equation

$$F_1 \cdots F_t = |F_1 \cdots F_t|, \qquad t \geq 2.$$

However, for $t > 2$, the solution reduces to that of systems of simultaneous algebraic equations which appear to be intractable. For $t = 2$, the complete solution is found easily by means of the algebra mentioned in Section 2.

With the notation as in (1.1), the complete solution of (4.1) is

$$F(n) \equiv \prod_i \frac{f(p_i) - 1}{f(p_i) - 2},$$

$$G(n) \equiv \prod_i [f(p_i)]^{s_i - 1} [f(p_i) - 1],$$

with $f(x)$ single valued and finite for prime values of x, $f(p) \neq 2$ for any prime p; and the like with the definitions of $F(n)$, $G(n)$ interchanged.

With suitable restrictions on f and the complex variable s, we therefore have

$$\sum_{n=1}^{\infty} \frac{F(n)}{n^s} \times \sum_{n=1}^{\infty} \frac{G(n)}{n^s} = \sum_{n=1}^{\infty} \frac{F(n)G(n)}{n^s} \quad ,$$

and if, for a particular f, anything were known about the singularities of the factor series on the left, Hadamard's theorem on the multiplication of singularities could be applied to the series on the right.

California Institute of Technology

LINEAR FUNCTIONALS AS DIFFERENTIALS OF A NORM

Robert C. James

FOREWORD. A differential is usually a linear function of the increment. The differential of the norm of a Banach space at a point x_0 is said to exist provided $D(x_0;y) = [D_t \|x_0 + ty\|]_{t=0}$ exists for each y. If it exists, $D(x_0;y)$ is a linear functional of the increment y. The purpose of this paper is to present a summary of what is known about the classes of Banach spaces for which each linear functional is a differential of the norm. For such spaces, this provides an easy method for establishing the form of a general linear functional. With the exception of Section 4, it is only assumed that the reader has a little familiarity with continuous additive functions.

1. *Fundamental concepts.* A *normed linear space* is a linear (vector) space [1, p. 26] whose multiplier domain F is the field of real numbers or the field of complex numbers and which has associated with each element x a real number $\|x\|$, the *norm* of x, satisfying the postulates:

1.1 $$\|x\| > 0 \quad \text{if } x \neq 0.$$

1.2 $$\|ax\| = |a| \; \|x\| \quad \text{for all numbers } a \text{ of } F.$$

1.3 $$\|x + y\| \leq \|x\| + \|y\|.$$

A normed linear space is said to be *complete* if for any sequence $\{x_n\}$ satisfying the Cauchy condition $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$ there is an element x for which $\lim_{n\to\infty} \|x - x_n\| = 0$. A complete normed linear space with real number multipliers will be called a *Banach space*. A complete normed linear space with complex number multipliers will be called a *complex Banach space*. The complex Banach spaces are a subset of the class of all Banach spaces, since multiplication in a complex Banach space can be restricted to multiplication by real numbers only.

A continuous function F with real number values and argument in a Banach space B is a *real linear functional*, or simply *linear functional*, if $f(x + y) = f(x) + f(y)$ for each x and y of B. A continuous function f with complex number values and argument in a complex Banach space B will be called a *complex linear functional* if $f(x + y) = f(x) + f(y)$ for each x and y of B and $f(ix) = if(x)$ for each x. Continuity of f is defined by $\lim_{n\to\infty} f(x_n) = f(x)$ if $\lim_{n\to\infty} \|x - x_n\| = 0$. The condition $f(x + y) \equiv f(x) + f(y)$ implies $f(ax) = af(x)$ for all real rational a and elements x, while continuity [and $f(ix) = if(x)$ if f is a complex linear functional] implies $f(ax) = af(x)$ for all a – real or complex according as f is a real or complex linear functional. If $f(x+y) \equiv f(x)+f(y)$ for a real or complex valued function f, then f is continuous if and only if it is continuous at $x=0$ [1, p. 23], or if and only if there exists a number M such that $|f(x)| \leq M\|x\|$ for each x [1, p. 54]. The least such number M is called the *norm of* f, written $\|f\|$.

Theorem 1.1. If F is a complex linear functional defined on a complex Banach space B, then $F(x) \equiv f(x) - if(ix)$, where f is the real linear functional which is the real part of $F(x)$ for each x. Likewise, if f is a real linear functional defined on a complex Banach space, then $F(x) \equiv f(x) - if(ix)$ defines a complex linear functional. In either case, $\|F\| = \|f\|$. Also, $|f(x)| = \|f\| \; \|x\|$ if and only if $|F(x)| = \|F\| \; \|x\|$ and $f(ix) = 0$.

Proof. Clearly any complex linear functional F is of the form $F(x) = f(x) + ig(x)$, where f and g are real linear functionals. But then $F(ix) = f(ix) + ig(ix)$, while $F(ix) = iF(x) = if(x) - g(x)$. Thus $g(x) = -f(ix)$ and $F(x) = f(x) - if(ix)$ for each x.

If f is a real linear functional defined on a complex Banach space, then F defined by $F(x) = f(x) - if(ix)$ is clearly continuous and satisfies $F(x+y) = F(x) + F(y)$. Also, $F(ix) = f(ix) - if(-x) = f(ix) + if(x) = iF(x)$. Thus F is a complex linear functional.

If $|f(x)| = \|f\| \; \|x\|$ for f a real linear functional on a complex Banach space, then $\|e^{i\theta}x\| = \|x\|$ and $|\cos\theta\, f(x) + \sin\theta\, f(ix)| \leq \|f\| \; \|x\|$ for all real θ. But $\theta = 0$ is an absolute maximum for $|\cos\theta\, f(x) + \sin\theta\, f(ix)|$, which is possible only if $f(ix) = 0$. Then $|F(x)| = |f(x)| = \|f\| \; \|x\|$ and $\|F\| \geq \|f\|$, where $F(x) \equiv f(x) - if(ix)$. But for any y, θ can be chosen so that $f(ie^{i\theta}y) = \cos\theta\, f(iy) - \sin\theta\, f(y) = 0$. Then $|F(y)| = |F(e^{i\theta}y)| = |f(e^{i\theta}y)| \leq \|f\| \; \|y\|$, so that $\|f\| \geq \|F\|$. Hence $\|F\| = \|f\|$. Conversely, if $|F(x)| = \|F\| \; \|x\|$, $f(ix) = 0$, and $\|F\| = \|f\|$, then clearly $|F(x)| = |f(x)| = \|f\| \; \|x\|$. This establishes the above theorem, except for cases in which there can exist a linear functional f with no x satisfying $|f(x)| = \|f\| \; \|x\|$. A limiting argument must then be used to show that $\|F\| = \|f\|$.

2. *Differentials of the norm.* Consider the ratio $[\|x+ty\| - \|x\|]/t$. If $t_1 > t_2 > 0$, then

$$[\|x+t_1y\| - \|x\|]/t_1 - [\|x+t_2y\| - \|x\|]/t_2 = [t_2\|x+t_1y\| - t_1\|x+t_2y\| + (t_1-t_2)\|x\|]/t_1t_2 \geq 0.$$

For it follows from 1.3 that $t_2\|x+t_1y\| - t_1\|x+t_2y\| = \|t_2x + t_1t_2y\| - \|t_1x + t_1t_2y\| \geq -(t_1-t_2)\|x\|$. But it also follows from 1.3 that

1. This theorem is also proved in [2], and for bilinear functionals in [6].

$[\ \|x + ty\| - \|x\|\ ]/t \geq -\|y\|$. Since this ratio has been shown to be a monotonically decreasing function of t, it follows that $\lim_{t\to+0}[\ \|x + ty\| - \|x\|\ ]/t = D_+(x;y)$ exists for each x and y.[2] Similarly, $\lim_{t\to-0}[\ \|x + ty\| - \|x\|\ ]/t = D_-(x;y)$ exists for each x and y.

Suppose that for a certain x and a linear functional f one has $f(x) = \|f\|\ \|x\|$. Subtracting this from $f(x \pm ty) \leq \|f\|\ \|x \pm ty\|$ gives $\pm tf(y) \leq \|f\|\ [\ \|x \pm ty\| - \|x\|\ ]$, and dividing by $\pm t$ gives (for $t > 0$):

$$2.1 \qquad \|f\| \left\{\frac{\|x - ty\| - \|x\|}{-t}\right\} \leq f(y) \leq \|f\| \left\{\frac{\|x + ty\| - \|x\|}{t}\right\}$$

Letting $t \to +0$, this becomes:

$$2.2 \qquad \|f\|\, D_-(x;y) \leq f(y) \leq \|f\|\, D_+(x;y),$$

for all x, y, and linear functionals f such that $f(x) = \|f\|\ \|x\|$. Proofs of 2.2 are also given in [7, p. 272] and [9, p. 75], where it is also shown that if $D_-(x;y) \leq a \leq D_+(x;y)$, then there is a linear functional f with $f(y) = a$, $f(x) = \|x\|$, and $\|f\| = 1$.

It will be said that the norm is *differentiable* at x if $\lim_{t\to0}[\ \|x + ty\| - \|x\|\ ]/t$ exists for each y. This limit will then be denoted by $D(x;y)$ and called the *differential of the norm* at x. $D(x;y)$ is also the derivative of $\|x + ty\|$ with respect to t. The following theorem is an immediate consequence of 2.2.

Theorem 2.1. Let x_0 be a non-zero element of a Banach space B. If the norm is differentiable at x_0, and $f(x_0) = \|f\|\ \|x_0\|$, then

$$\|f\|\, D(x_0;y) = f(y)$$

for each y of B.

Theorem 2.2. If $D(x_0;y)$ exists for an element x_0 of a Banach space B, then $D(x_0;y)$ is a linear functional of y with $\|D\| = 1$ and $D(x_0;x_0) = \|x_0\|$.

If one uses the theorem [1, p. 55] that for each x of a Banach space B there is a linear functional f with $\|f\| = 1$ and $f(x) = \|x\|$, then Theorem 2.2 is an immediate consequence of Theorem 2.1. It can also be proved directly: Note first that $|(\ \|x + ty\| - \|x\|\ )/t| \leq \|y\|$ follows from 1.3. Thus $|D(x;y)| \leq \|y\|$ for each y, while $D(x;x) = \|x\|$. Hence $\|D\| = 1$. It only remains to show that $D(x;y + z) = D(x;y) + D(x;z)$

[2] Other proofs are given in [9, p.75], [12, Theorem 5.21], and [7, p.272].

for all y and z. This is equivalent to $\lim_{t\to 0}[\,\|x + t(y + z)\| + \|x - ty\| + \|x - tz\| - 3\|x\|\,]/t = 0$. But this limit must be zero, since by successive use of 1.3 one gets $\|x + t(y + z)\| + \|x - ty\| + \|x - tz\| - 3\|x\| \geq \|2x + tz\| + \|x - tz\| - 3\|x\| \geq \|3x\| - \|3x\| = 0$.

3. *Uniformly convex Banach spaces.* Suppose that $D(x;y)$ exists for each $x \neq 0$ of a Banach space B. If for a linear functional f of B there is a non-zero element x_0 with $f(x_0) = \|f\|\|x_0\|$, then it follows from Theorem 2.1 that $f(y) \equiv \|f\|\, D(x_0;y)$. Moreover, from Theorem 2.2 one sees that, for each x_0, $kD(x_0;y)$ is a linear functional with norm equal to $|k|$. Thus if for each linear functional f there is an x_0 with $f(x\ ) = \|f\|\ \|x_0\|$, then the set of linear functionals consists precisely of all multiples of differentials of the norm.

A Banach space is said to be *uniformly convex* [3] if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|x + y\| < 2 - \delta$ whenever $\|x\| = \|y\| = 1$ and $\|x - y\| > \epsilon$. The *unit sphere* of a Banach space is the set of all x for which $\|x\| \leq 1$. The *surface of the unit sphere* is the set of all x for which $\|x\| = 1$. If x and y are on the surface of the unit sphere of a uniformly convex Banach space, then the mid-point $\frac{1}{2}(x + y)$ of the line joining x and y must be in the interior of the unit sphere and can be close to the surface of the sphere only if $\|x - y\|$ is small. The proof of the following well-known theorem is suggested by this and the concept of a *hyperplane* as the set of all x satisfying $f(x) = c$ for a fixed linear functional f and constant c. The hyperplane consisting of all x for which $f(x) = \|f\|$ is at zero distance from the unit sphere, and is a tangent hyperplane to the unit sphere if there is an element x with $f(x) = \|f\|$ and $\|x\| = 1$.

Theorem 3.1. *If the norm of a uniformly convex Banach space B is differentiable at each non-zero point x, then for each linear functional f defined on B there is a unique element x_0 such that $\|x_0\| = 1$ and*

$$f(y) \equiv \|f\|\, D(x_0;y).$$

If B is also a complex Banach space, then for each complex linear functional F there is a unique element x_0 such that $\|x_0\| = 1$ and

$$F(y) \equiv \|F\|\,[D(x_0;y) - iD(x_0;iy)].$$

Proof. If it is shown that for each linear functional f there is a unique x_0 with $\|x_0\| = 1$ and $f(x_0) = \|f\|$, then it will follow from Theorems 2.1 and 2.2 that this is the unique x_0 such that $\|x_0\| = 1$ and $f(y) \equiv \|f\|\, D(x_0;y)$. But $\|f\|$ is the l.u.b. of $f(x)$ for $\|x\| = 1$. Hence there is a sequence $x_1, x_2, \cdots$ of elements such that $\|x_n\| = 1$

for each n and $\lim_{n\to\infty} f(x_n) = \|f\|$. For an arbitrary $\epsilon > 0$, choose a $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| > \epsilon$ imply $\|x + y\| < 2 - \delta$. Choose N such that $f(x_n) > \|f\|(1 - \frac{1}{2}\delta)$ if $n > N$. If $n > N$ and $m > N$, then $f(x_n + x_m) = f(x_n) + f(x_m) > 2\|f\|(1 - \frac{1}{2}\delta)$. Also, $f(x_n + x_m) \leq \|f\| \|x_n + x_m\|$ because of the definition of $\|f\|$. Hence $\|x_n + x_m\| > 2 - \delta$, and $\|x_n - x_m\| \leq \epsilon$. The sequence $\{x_n\}$ thus satisfies the Cauchy condition. Since B is complete, there is an x for which $\lim_{n\to\infty} \|x - x_n\| = 0$. Then $\lim_{n\to\infty} f(x - x_n) = 0$, or $f(x) = \|f\|$. Clearly $\|x\| = 1$.

Suppose B is also a complex Banach space and F is a complex linear functional on B. Let f be the real part of F and choose x_0 as above so that $f(y) \equiv \|f\| D(x_0; y)$. It then follows from Theorem 1.1 that $F(y) \equiv \|f\| [D(x_0; y) - iD(x_0; iy)]$.

4. *Other classes of Banach spaces.* Other assumptions can be used in place of uniform convexity to give conclusions such as those of Theorem 3.1. Some of these will be discussed briefly here, most proofs being omitted with references given.

The unit sphere of a Banach space B is said to be *weakly compact* if for any sequence of elements $\{x_n\}$ in the unit sphere there is a subsequence $\{x_{p_n}\}$ and an element x such that $\{x_{p_n}\}$ *converges weakly* to x in the sense that $\lim_{n\to\infty} f(x_{p_n}) = f(x)$ for each linear functional f defined on B. But if, for a certain f, $\{x_n\}$ is chosen so that $\|x_n\| = 1$ for each n and $\lim_{n\to\infty} f(x_n) = \|f\|$, then it follows that $f(x) = \|f\|$. Also, $\|x\| = 1$. For $|f(x)| \leq \|f\| \|x\|$ and $f(x) = \|f\|$ imply $\|x\| \geq 1$. But there is known [1, p. 55] to be a linear functional g satisfying $|g(x)| = \|g\| \|x\|$. Then $|g(x_n)| \leq \|g\|$ and $\lim_{n\to\infty} g(x_n) = g(x)$ imply $\|x\| \leq 1$. Thus the following theorem is an immediate consequence of of Theorem 2.1.

Theorem 4.1. *If the unit sphere of a Banach space B is weakly compact and the norm of B is differentiable for each non-zero x, then for each linear functional f defined on B there is at least one element x_0 such $\|x_0\| = 1$ and $f(y) \equiv \|f\| D(x_0; y)$.*

Any uniformly convex Banach space has a weakly compact unit sphere [10, Theorems 1 and 2], though not every Banach space with a weakly compact unit sphere is uniformly convex – and in fact may not be uniformly convex for any norm equivalent to the given one [4]. Thus the assumption of uniform convexity in Theorem 3.1 is stronger than the assumption of weak compactness in Theorem 4.1.

The norm of a Banach space B is said to be *uniformly Fréchet differ-*

entiable (or *uniformly strongly differentiable*) if $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for each x and y and the limit is approached uniformly for $\|x\| = 1$ and $\|y\| \leq 1$. That is, for any $\epsilon > 0$ there is a $\delta > 0$ such that if $\|x\| = 1$ and $\|y\| \leq 1$ then

$$\left| \frac{\|x + ty\| - \|x\|}{t} - D(x;y) \right| < \epsilon, \text{ or equivalently,}$$

$$\left| \frac{\|x + ty\| - \|x\|}{t} - D(x;y) \right| < \epsilon \|y\| .$$

The *first conjugate space* B^* of a Banach space B is the Banach space of all linear functionals defined on B, with $\|f\|$ as the norm [1, p. 188]. A Banach space is said to be *reflexive* if for any linear functional F defined on B^* there is an element x_F of B for which $F(f) = f(x_F)$ for each f of B^*. It is easily shown [1, pp. 189-190] that, for a reflexive space, $F \to x_F$ defines an isometry (a norm-preserving isomorphism) between B and B^{**}. Then any linear functional f defined on B can be regarded also as a linear functional f' defined on B^{**}, $f(x_F) = F(f)$ being equivalent to $f'(F) = F(f)$ for each F of B^{**}. Thus B is reflexive if and only if B^* is reflexive. It is known that the unit sphere of a Banach space B is weakly compact if and only if B is reflexive [5, p. 53]. Thus if B^* is uniformly convex, then the unit spheres of B^* and of B are weakly compact. But it is known that B^* is uniformly convex if and only if the norm of B is uniformly Fréchet differentiable [11, p. 647]. Thus the following theorem is a consequence of Theorem 4.1.

Theorem 4.2. *If the norm of a Banach space B is uniformly Fréchet differentiable, then for each linear functional f defined on B there is at least one element x_0 such that $\|x_0\| = 1$ and $f(y) \equiv \|f\| D(x_0;y)$.*

In neither Theorem 4.1 nor Theorem 4.2 can one conclude the uniqueness of the element x_0. It is not necessary to assume uniform convexity for this, but sufficient to know that the space is *strictly convex*. That is, that whenever $\|x\| = \|y\| = 1$ and $\|x - y\| \neq 0$, then $\|x + y\| < 2$. For if $f(x) = \|f\|$ and $f(y) = \|f\|$ with $\|x\| = \|y\| = 1$, then $|f(x + y)| \leq \|f\| \|x + y\|$ and $f(x + y) = 2\|f\|$. Thus $\|x + y\| = 2$ and $x = y$ if the space is strictly convex. Strict convexity is a great deal weaker than uniform convexity, since any Banach space can be given an equivalent norm for which the space is strictly convex [3, Theorem 9].

It should be noted that whenever the form of real linear functionals is known for a complex Banach space, the form of complex linear functionals can be determined by use of Theorem 1.1. Thus the conclusion of Theorem 3.1 for complex linear functionals could also be made for

Theorems 4.1 and 4.2, except for the uniqueness of x_0.

5. *An application.* Let Hilbert space be defined as a Banach space for which there is associated with each pair of elements x,y an *inner product* (x,y) which is a real number satisfying the postulates:

5.1 $$(x,x) = \|x\|^2,$$

5.2 $$(x, y+z) = (x,y) + (x,z),$$

5.3 $$(x, ty) = t(x,y),$$

5.4 $$(x,y) = (y,x).$$

Then $\{\|x+ty\| - \|x\|\}/t = \{[(x+ty, x+ty)]^{\frac{1}{2}} - [(x,x)]^{\frac{1}{2}}\}/t = \{[(x,x) + 2t(x,y) + t^2(y,y)]^{\frac{1}{2}} - [(x,x)]^{\frac{1}{2}}\}/t = [2(x,y) + t\|y\|^2]/[\|x + ty\| + \|x\|]$. The limit of this as t approaches zero is clearly uniform for all x and y with $\|x\| = 1$ and $\|y\| \leq 1$ and is equal to $(x,y)/\|x\|$ whenever $\|x\| \neq 0$. Thus it follows from Theorem 4.2 that for each linear functional f defined on Hilbert space there is an element x_0 such that $f(y) \equiv (\|f\|/\|x_0\|)(x_0,y)$. Letting $x_0' = (\|f\|/\|x_0\|)x_0$, this becomes

$$f(y) \equiv (x_0', y).$$

It is also not difficult to show that Hilbert space is uniformly convex. For $(x+y, x+y) = 2[(x,x) + (y,y)] - (x-y, x-y)$ follows from 5.1-5.4 and is equivalent to $\|x+y\| = 2 - [2 - (4 - \|x-y\|^2)^{\frac{1}{2}}]$ if $\|x\| = \|y\| = 1$. Thus if $\|x-y\| > \epsilon$, then $\|x+y\| < 2 - \delta$ if $\delta = 2 - (4 - \epsilon^2)^{\frac{1}{2}}$.

A *complex Hilbert space* is a complex Banach space having a complex valued inner product satisfying 5.1, 5.2, and

5.3′ $$(x, ty) = \bar{t}(x,y),$$

5.4′ $$(x,y) = \overline{(y,x)}.$$

But a complex Hilbert space is a real Hilbert space with an inner product equal to the real part of (x,y). Thus it follows from Theorem 1.1 and the above that for any complex linear functional F there is a unique element x_0 such that $F(y) \equiv R[(x_0,y)] - iR[(x_0, iy)]$. But $R[(x_0, iy)] = R[-i(x_0,y)]$ is the imaginary part of (x_0,y), so that

$$F(y) \equiv \overline{(x_0,y)} \equiv (y, x_0).$$

Further applications of Theorem 3.1 are also given in [8] and [7, pp. 289-291].

REFERENCES

1. S. Banach, *Théorie des operations linéaires*, Warsaw, 1932.

2. H. F. Bohnenblust and A. Sobczyk, *Extensions of functionals on complex-linear*

spaces, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 91-93.
3. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 396-414.
4. M. M. Day, *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313-317.
5. W. F. Eberlein, *Weak compactness in Banach spaces I*, Proc. Nat. Acad. Sci. vol. 33 (1947) pp. 51-53.
6. M. Fréchet, *Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espaces de Hilbert*, Ann. of Math. vol. 36 (1935) pp. 705-718.
7. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 265-292. This is essentially the author's California Institute of Tech. doctoral thesis (1945) written under the direction of Professor A. D. Michal.
8. E. J. McShane, *Linear functionals on certain Banach spaces*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 402-408.
9. S. Mazur, *Über convexe Mengen in linearen normierten Räumen*, Studia Mathematica vol. 4 (1933) pp. 70-84.
10. D. Milman, *On some criteria for the regularity of spaces of the type (B)*, C. R. (Doklady) Acad. Sci. URSS Vol. 20 (1938) pp. 243-246.
11. V. Smulian, *Sur la dérivabilité de la norme dans l'espace de Banach*, C. R. (Doklady) Acad. Sci. URSS vol. 27 (1940) pp. 643-648.
12. A. E. Taylor, *Derivatives in the calculus*, Amer. Math. Monthly vol 49 (1942) pp. 631-642.

University of California

COLLEGIATE ARTICLES

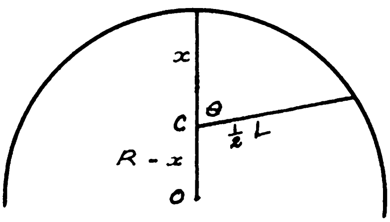
Graduate training not required for Reading

A GENERALIZATION OF BUFFON'S NEEDLE PROBLEM IN PROBABILITY

W. Funkenbusch

A rod of length L is tossed on a washer of radii $R \pm \frac{1}{2} L$. Determine the ratio of the rod length to the ring diameter such that the probability of rod-ring contact when the center of the rod falls on the washer is the same whether the center of the rod falls inside or outside or R.

1. Assume the center of the rod is inside of R. Let C be the center of the rod, O the center of the ring, θ the angle between the rod and the line from C normal to the ring, and x the distance from C to the ring. We have below



Solving for x by the law of cosines gives

$$(1.1)\qquad x = \frac{L\cos\theta + 2R \pm \sqrt{4R^2 - L^2\sin^2\theta}}{2}$$

$$(1.2)\qquad \therefore\ 4R^2 - L^2\sin^2\theta \geq 0 \quad \text{or} \quad R \geq \tfrac{1}{2}L\sin\theta$$

$$(1.3)\qquad \text{and since}\quad 0 \leq \theta \leq \tfrac{1}{2}\pi,\quad R \geq \tfrac{1}{2}L$$

$$(1.4)\qquad \text{Now}\quad x \leq \tfrac{1}{2}L \leq R \leq \tfrac{1}{2}L\cos\theta + R$$

∴ Root with negative sign before radical is only possible one. Then, rod will cut ring if

$$(1.5)\qquad x < \frac{L\cos\theta + 2R - \sqrt{4R^2 - L^2\sin^2\theta}}{2}$$

and should the center of the rod fall on the washer inside of R, the probability that contact is achieved is

(1.6) $$P_i = \frac{\displaystyle\int_0^{\frac{\pi}{2}} \left[\frac{L\cos\theta + 2R - \sqrt{4R^2 - L^2\sin^2\theta}}{2}\right] d\theta}{\frac{L}{2}\cdot\frac{\pi}{2}}$$

(1.7) $$= \frac{2}{\pi} + \frac{1}{k} - \frac{2}{k\pi}\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\, d\theta \quad \text{where } k = \frac{L}{2R}$$

or in terms of elliptic integrals then

(1.8) $$P_i = \frac{2k + \pi - 2E}{k\pi}$$

2. Let $\overline{P_i}$ be the probability that the center of the rod falls on the washer inside of R. Clearly then

(2.1) $$\overline{P_i} = \frac{\pi[R^2 - (R - \tfrac{1}{2}L)^2]}{\pi[(R + \tfrac{1}{2}L)^2 - (R - \tfrac{1}{2}L)^2]} = \frac{2 - k}{4}$$

3. For C exterior to circle O, rod *from* C to the circle equal to $\tfrac{1}{2}L$, and distance from C to O equal to $R + |x|$, we have

(3.1) $$x = \frac{L\cos\theta - 2R \pm \sqrt{4R^2 - L^2\sin^2\theta}}{2}$$

from (1.3) $R \geq |\tfrac{1}{2}L|$ then

(3.2) $$R \geq |\tfrac{1}{2}L| \geq |\tfrac{1}{2}L\cos\theta|$$

(3.3) $$|\tfrac{1}{2}L\cos\theta| - R \leq 0$$

and since $x \leq 0$ and $|x| \leq |\tfrac{1}{2}L| \leq R$, the root with the positive sign before the radical is the only possible one. Then the rod will cut the ring if

(3.4) $$|x| < \left|\frac{L\cos\theta - 2R + \sqrt{4R^2 - L^2\sin^2\theta}}{2}\right|$$

and should the center of the rod fall on the washer outside of R, the probability that contact is achieved is

(3.5) $$P_i = \frac{\displaystyle\int_0^{\frac{\pi}{2}} \left[\frac{L\cos\theta - 2R + \sqrt{4R^2 - L^2\sin^2\theta}}{2}\right] d\theta}{\frac{L}{2}\cdot\frac{\pi}{2}}$$

(3.6) $$P_0 = \frac{2k - \pi + 2E}{k\pi}$$

4. Let $\overline{P_0}$ be the probability that the center of the rod falls on the washer outside of R. Clearly then

(4.1) $$P_0 = \frac{\pi[(R + \tfrac{1}{2}L)^2 - R^2]}{\pi[(R + \tfrac{1}{2}L)^2 - (R - \tfrac{1}{2}L)^2]} = \frac{2 + k}{4}$$

5. By definition of problem

(5.1) $$P_i\overline{P_i} = P_0\overline{P_0} \quad \text{which gives}$$

(5.2) $$E = \frac{\pi - k^2}{2} \quad \text{and since}$$

(5.3) $$E = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 \frac{k^{2n}}{2n - 1}\right\}$$

equating right sides of (5.2) and (5.3)

(5.4) $$k^2\left\{1 - \pi \sum_{n=1}^{\infty}\left[\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 \frac{k^{2n-2}}{2n - 1}\right\} = 0$$

The straight line case (*i.e.* Buffon's needle problem) is given by the solution $k = 0$.
The other solution is given by the equation

(5.5) $$\sum_{n=1}^{\infty}\left[\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 \frac{k^{2n-2}}{2n - 1} = \frac{1}{\pi}$$

By the Ratio Test the left side of (5.5) converges if $k^2 < 1$, and therefore for our purpose $k < 1$ from (1.3) and (1.7). By Descartes' Rule of Signs (5.5) has but one positive real root for k^2, and since we take k as being positive, only one root for k.
Since we then have a unique constant, let us denote it by γ, and define it by the equation

(5.6) $$\sum_{n=1}^{\infty}\left[\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 \frac{\gamma^{2n-2}}{2n - 1} = \frac{1}{\pi}$$

Also by definition of problem then $\gamma = \frac{L}{2R} = \frac{L}{D}$ or that is it represents the ratio between rod length and ring diameter for equally probable inside-outside rod-ring contact in a contactually possible field.

The four probabilities in the problem then can be expressed as functions of γ

(5.7)
$$P_i = \frac{2+\gamma}{\pi}$$
$$\overline{P_i} = \frac{2-\gamma}{4}$$
$$P_o = \frac{2-\gamma}{\pi}$$
$$\overline{P_o} = \frac{2+\gamma}{4}$$

The solution for γ may be approximated by using a table of elliptic integrals and approximating the non-trivial solution for k in equation (5.2).

(5.2)
$$E = \frac{\pi - k^2}{2}$$

$\sin^{-1} k$		k		E		right side (5.2)
66.0°	----	.9135	----	1.1546	>	1.1534
66.4°	----*	.9164 .9164	----	**1.1508	~	1.1509
66.5°	----	.9171	----	1.1499	<	1.1552

*interpolated and table value agree
**interpolated value

∴ It would appear that a close fit for γ is $\gamma \sim 0.9164$.

Michigan College of Mining and Technology
Sault Ste. Marie Branch

FINITE GROUPS GENERATED BY INVOLUTIONS ON A LINE

Alan Wayne

1. *Introduction.* We may think of a projectivity on a line as the effect of an operator P_i which transforms a coordinate z so that

$$(1) \qquad P_i z = (a_i z + b_i)/(c_i z + d_i), \quad (a_i d_i - b_i c_i \neq 0).$$

The continued application of any set of these operators generates a subgroup of the projective group on a (real or complex) line.

When the projectivities are involutory, that is, of period two, the particular subgroups generated are groups of *involutions on a line*. In this case, $P_i^2 z = z$, which is true if and only if $d_i = -a_i$, so that an involution on a line is defined by

$$(2) \qquad L_i z = (a_i z + b_i)/(c_i z - a_i), \quad (a_i^2 + b_i c_i \neq 0).$$

Two kinds of involutions on a line which have been objects of study are the operations of *subtraction* and *division*, defined respectively by $S_h z = h - z$, and $D_k z = k/z$. The operators S_h and D_k, separately or together, generate subgroups of the groups of involutions on a line.

All of the groups mentioned above are, in general, infinite groups. However, necessary and sufficient conditions that a *finite* group be generated by S_h and D_k have been given by J. S. Frame [4], who extended the results given in earlier papers by E. J. Finan [2], and G. A. Miller [5]. It was shown by Finan that the complex numbers h and k must be such that $x = h^2/k$ is a zero of one of the polynomials:

$$(3) \qquad \begin{aligned} &x^m - \binom{2m-1}{1} x^{m-1} + \binom{2m-2}{2} x^{m-2} - \cdots + (-1)^m \binom{m}{m}, \\ &x^{m-1} - \binom{2m-2}{1} x^{m-2} + \binom{2m-3}{2} x^{m-3} - \cdots + (-1)^{m-1} \binom{m}{m-1}. \end{aligned}$$

It was further shown by Frame that the above condition is equivalent to

$$(4) \qquad x = h^2/k = 4 \cos^2 s\pi/n = 2\cos(2s\pi/n) + 2,$$

where s and n are coprime integers.

In the present paper we shall further extend the above results by finding the necessary and sufficient condition under which a finite group is generated by operators L_1 and L_2 corresponding to any two two involutions on a line.

2. *The condition for finiteness.* Since we have $L_1^2 z = L_2^2 z = z$, we need examine only those products in which L_1 and L_2, whether distinct

or not, alternate as factors. There are, then, using left multiplication, only four types of products: $(L_2L_1)^a z$, $(L_1L_2)^b z$, $(L_2L_1)^c L_2 z$, and $(L_1L_2)^d L_1 z$, where the exponents are non-negative integers. By equating these products in pairs in all of the sixteen possible ways, it is found that L_1 and L_2 generate a finite group if and only if there exists a positive integer n such that $(L_2L_1)^n z = z$.

We now make use of the fact [3, page 629] that a projectivity $P_i z$ is of period n if and only if

$$(5) \qquad (a_i + d_i)^2 = 4(a_i d_i - b_i c_i)\cos^2 s\pi/n,$$

where s and n are coprime integers. Let the projectivity under consideration be $(L_2L_1)z$. Then L_1 and L_2 generate a finite group if and only if

$$(6) \qquad u^2/v = 2 + 2\cos(2s\pi/n),$$

where $u = b_1c_2 + 2a_1a_2 + b_2c_1$, and $v = (a_1^2 + b_1c_1)(a_2^2 + b_2c_2)$. Thus we have shown that Frame's result (2) applies actually to any two involutions on a line.

We observe that $w^n = 1$ if and only if it is also true that $w = \cos(2s\pi/n) + i\sin(2s\pi/n)$, where $i = \sqrt{-1}$. But, from (6), by a computation, it is found that $w = (u - \sqrt{u^2 - 4v})/(u + \sqrt{u^2 - 4v})$. Thus we have the following theorem.

Theorem 1. *The operators L_1 and L_2 generate a finite group if and only if $w^n = 1$, where $w = (u - \sqrt{u^2 - 4v})/(u + \sqrt{u^2 - 4v})$, $u = b_1c_2 + 2a_1a_2 + b_2c_1$, and $v = (a_1^2 + b_1c_1)(a_2^2 + b_2c_2)$.*

3. *The* (1,2,3) *theorem.* There is a remarkable consequence of Theorem 1. Let $a_2 = -1$, $c_2 = 1$, $a_2 = c_1 = 0$, and $b_1 = b_2 = t$, where t is an integer. Then $u = v = t$, and the resulting line involutions are the very special subtractions and divisions corresponding to S_t and D_t. Since S_t and D_t are distinct, $t^2 - 4t \neq 0$, so that t is neither zero nor four. Moreover, since $w \neq 1$, it follows that $t^2 < 4t$, in order to have $w^n = 1$. Hence $t = 1$, 2, or 3.

The result is the following beautiful theorem, first noted by G. A. Miller [5, page 84].

Theorem 2. *The only integers t for which S_t and D_t generate a finite group are* 1, 2, *and* 3.

4. *The nature of the finite groups.* By virtue of the relations $L_1^2 z = L_2^2 z = (L_2L_1)^n z = z$, where L_2L_1 is of proper order n, it is seen that the finite groups generated by L_1 and L_2 are the *dihedral groups*, of order $2n$, otherwise known as the groups of symmetries of the plane

regular polygons [1, page 181]. The properties of the equation $w^n = 1$ lead to verification of a result of G. A. Miller [5, page 86], namely: there will always exist, for a given n, at least one set of values of u and v such that L_1 and L_2 generate a dihedral group.

In the real inversive plane, the geometric significance of the groups of subtraction and division, as special dihedral groups defined by (4), has been described in detail by Sister M. Philip Steele and V. O. McBrien [7].

The groups generated by S_t and D_t, for $t = 1$, 2, and 3, are respectively, the groups of symmetries of the equilateral triangle, square, and regular hexagon. The first of these is more celebrated as the "cross ratio group."

Applications of these last three special groups to elementary trigonometry have been made by G. A. Miller [6].

REFERENCES

1. Carmichael, R. D. *Introduction to the Theory of Groups of Finite Order.* Ginn and Company, New York, 1937.
2. Finan, E. J. "On groups of subtraction and division," *American Mathematical Monthly*, 1941, Vol. 48, pp. 3-7.
3. Forsyth, A. R. *Theory of Functions of a Complex Variable.* Cambridge University Press, 1918.
4. Frame, J. S. "Note on groups of subtraction and division," *American Mathematical Monthly*, 1941, Vol. 48, pp. 468-469.
5. Miller, G. A. "Groups of subtraction and division," *Quarterly Journal of Pure and Applied Mathematics*, 1906, Vol. 37, pp. 80-87.
6. Miller, G. A. "A new chapter in trigonometry," *Quarterly Journal of Pure and Applied Mathematics*, 1906, Vol. 37, pp. 226-234.
7. Steele, Sister M. Philip and McBrien, Vincent O. "Basic configurations of the plane under certain groups" *Mathematics Magazine*, 1949, Vol. 23, pp. 5-14.

ADVANCED ALGEBRA: OPERATIONS WITHOUT NUMBERS

Richard Arens

Algebra begins when it is recognized in arithmetic that there are laws, such as

$$a + b = b + a \tag{1}$$

or

$$(ab)c = a(bc), \tag{2}$$

which are satisfied by the *real numbers* but which can be expressed without the use of numbers (although '*a*', '*b*', and '*c*' in (1), (2) may be replaced by the signs (names) of numbers; and (1) is different from the primitive statement: "See how 1 + 2 = 2 + 1, 3 + 6 = 6 + 3, 2 + 5 = 5 + 2, and thus it goes.")

Advanced algebra goes beyond the formulation and proof of laws obeyed by the real numbers. It considers that there may be classes of things *not numbers* (but such as *vectors, matrices, . . .*) in which laws like (1) and (2) make sense and perhaps hold or fail, as the case may be.

We turn at once to an imaginary situation out of which a class of things, not numbers, with operations and laws will be constructed.

Imagine a class of chairs, say five. Let us refer to each chair by one of the Greek letters α, β, γ, δ, ϵ. Let one man occupy each chair. Now let them all arise, shuffle around, and reseat themselves in a definite way, which we shall call p. Now what do we mean by a "definite way p"?

Suppose that in this reseating p the man in α goes to δ, the man in β goes to α, the man in γ goes to β, the man in δ goes to ϵ, the man in ϵ goes to γ. Then we say that

$$p = \begin{bmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \delta & \alpha & \beta & \epsilon & \gamma \end{bmatrix}, \tag{3}$$

where the symbol on the right gives a convenient abbreviation for the *permutation* described in the preceeding chairs. A different permutation might take the form

$$q = \begin{bmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \delta & \gamma & \alpha & \beta & \epsilon \end{bmatrix} \tag{4}$$

Note that here the man in ϵ sat down again in his old place. We didn't exclude that. These physical events might be described at length as "permutations of men who are seated in the chairs α, β, γ, δ, ϵ," but it is convenient to call them "permutations of α, β, γ, δ, ϵ." Notice that although the men do the moving, it is not necessary or helpful to assign names or symbols to them; everything can be described by means of the chairs, which are not moved.

These permutations are in fact *functions*, for we may define

(5) $p(\alpha) = \delta$, $p(\beta) = \alpha$, $p(\gamma) = \beta$, $p(\delta) = \epsilon$, $p(\epsilon) = \gamma$.

Then p is a function whose domain is the class α, β, γ, δ, ϵ and whose range or class of values is α, β, γ, δ, ϵ, with the added condition:

(6) for every element ζ of α, β, γ, δ, ϵ there is one and no more than one element η whose p-value is ζ, that is $p(\eta) = \zeta$. Thus *function* p is what we mean by "the definite way of reseating, p."

In (6), ζ and η are *variables*, for example, if we let $\zeta = \gamma$ then $\eta = \epsilon$ as one can see from (5) or (4).

To save words and letters, let us denote the class of chairs (or whatever the class is on which the permutations act) by Γ, so that $\Gamma = \{\alpha,\beta,\gamma,\delta,\epsilon\}$. And rather than saying, for example, "β is an element of Γ" or "β belongs to Γ" or "β is a member of Γ" we shall write "$\beta\epsilon\Gamma$." This symbol 'ϵ' is widely used in such a sense.*

Now whenever you have two functions p and q (as such permutations) where the *range* (class of "outputs") of one, say p, is the *domain* (class of "inputs") of the other, q, (as in the case of permutations, where in fact all ranges and domains are Γ) you can *substitute* $p(\zeta)$ for each $\zeta\epsilon\Gamma$ into q, obtaining a *new* permutation r, where

(7) $$r(\zeta) = q(p(\zeta)) \quad (\text{for every } \zeta\epsilon\Gamma).$$

By following through (3) and (4), one arrives at

(8) $r(\alpha) = \beta$, $r(\beta) = \delta$, $r(\gamma) = \gamma$, $r(\delta) = \epsilon$, $r(\epsilon) = \alpha$.

This can be written also as

(9) $$r = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \beta & \delta & \gamma & \epsilon & \alpha \end{pmatrix}.$$

One see that (6) holds for r, proving that r *is* a permutation, as could be expected from (7) and (6), *without* the inspection of (8) or (9).

Now we need a symbol to represent the fact that r is the result of substituting p into q. The symbol, which is widely used, is a little circle '$\circ$' and we write

(10) $$r = q \circ p.$$

That is to say, (10) is an abbreviation for (7), and its advantage lies in the avoiding of all mention of the variable ζ in (7).

With a sideward glance at p we direct our attention to a permutation s given by

(11) $$s = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \beta & \gamma & \epsilon & \alpha & \delta \end{pmatrix}$$

*See the outside cover of the text-book referred to at the end of this article.

or alternatively (in the manner of (8))

$$s(\alpha) = \beta, \quad s(\beta) = \gamma, \quad s(\gamma) = \epsilon, \quad s(\delta) = \alpha, \quad s(\epsilon) = \delta,$$

or in terms of variables

(12) $\eta = p(\zeta)$ if and only if $\zeta = s(\eta)$.

Constructing the scheme for $s \circ r$ we get

$$s \circ r = \begin{bmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \alpha & \beta & \gamma & \delta & \epsilon \end{bmatrix} .$$

The result is a permutation which doesn't really permute anybody. We thus have to admit this "neutral" permutation as a *bona fide* permutation (some call it the "identical" or "identity" transformation) or we should not be able to assert for *any* permutations p and q of Γ that $p \circ q$ is a permutation. Next, (11) shows that given p we can find s such that $s \circ p$ is the neutral permutation. Just as '$\circ$' is always used to denote substitution, so we shall denote the s constructed in (11) by $\overset{-1}{p}$. Thus $\overset{-1}{p} \circ p$ or $p \circ \overset{-1}{p}$ is the neutral permutation (the *latter* follows from (12)).

Now consider *any* class G of elements such that there is defined for each pair $a \in G$, $b \in G$ a binary operation $a \top b$ (the sign is a sawed-off '+' sign) having the properties

(13) $(a \top b) \top c \quad a \top (b \top c)$ for $a, b, c \in G$;

(14) there is an element e in G such that

$$e \top a = a \text{ for every } a \in G;$$

(15) for every $a \in G$ there is a $b \in G$ such that $b \top a = e$.

Such a class of things, G, is called a *group*. The origin of the word is this: before the abstract axioms (13), (14), (15) were studied on their own account, much work had been done for groups of permutations and groups of transformations. In the abstract case, the things may not be transformations, so the words "of transformations" were just dropped off. We have seen how (13), (14), (15) may be verified for any group of permutations.

The axioms do not prohibit G's containing an infinite number of elements. If G contains only a finite number of elements that number is called the order of G. The group of *all* permutations of a class Γ of n things has order $= n!$, as one learned in elementary algebra. But one could have a group S of permutations of Γ which was not the whole group of *all* permutations. For example, if p be the permutation described in (3), then

$$S = p,\ p \circ p,\ p \circ p \circ p,\ p \circ p \circ p \circ p,\ p \circ p \circ p \circ p \circ p \ (\S)$$

(§) The parentheses grouping the factors may be omitted since there is no ambiguity, by (13).

is a *subgroup* of the group of all permutations of Γ. To see this, the reader (student) will have to compute the symbols (such as (3)) for all the elements of S, and then verify (13), (14), (15). (13) is clear; for (14) compute the last element of S and you will find it is the neutral permutation; (15) follows from the previous remark.

Now for groups of *permutations* certain consequences of (13), (14), (15) are obvious.

(16) There is only one element e satisfying (14), and it has the property $a \top e = a$ also.

(17) There is only one element b such that $b + a = e$, and for it $a \top b = e$ also holds.

However, (16) *and* (17) *are consequences of the group axioms* (13), (14), (15).

Proof: (Let us drop the sign '$\top$' and write ab for $a \top b$). Suppose $ba = e$. Find c such that $cb = e$. Then $ab = eab = cbab = ceb = cb = e$. So whenever $ba = e$ then also $ab = e$. Moreover, $ae = aba = ea = e$.

Since the b for which $ba = e$ has now been shown to be unique we are entitled to call b *the inverse* of a, and to denote it by a^{-1}, so that $a \top a^{-1} = a^{-1} \top a = e$. We are also entitled to call e *the neutral* element of G.

We computed $r = q \circ p$ in (9). We invite the reader to compute $p \circ q$. It turns out to be a permutation different from r. It must therefore not be expected to be derivable from the axioms of groups that $a \top b = b \top a$ for every a and b. If $a \top d = d \top a$ in a group, a and d are said to *commute*. Thus a and the neutral element e, as well as a and a^{-1}, commute. If every two elements in a group G commute, G is called *abelian* (after the mathematician Abel).

An example of an abelian group is the group of real numbers R, where $a \top b$ means $a + b$. Here $-a$ is a , and 0 is the neutral element. One might suppose that the real numbers form a group if one sets $a \top b = a \times b$. (13) and (14) are surely satisfied, if we take 1 as the neutral element. However, since $b \times 0 = 0 = 1$ for every b, 0 has no inverse whence R is not a group under '$\times$'. However, if we consider only the non-zero real numbers R_0, they *do* form a group under '$\times$'. Even the positive real numbers R_+ form a group (under '$\times$'). It is a subgroup of R_0. Finally the subgroup S defined for illustrative purposes above is abelian.

We turn now to an important situation which may arise when two groups G and H are considered. Let us use '$\top$' and '$\cdot$' as the operations in G resp. H; and let us use 'e' and '1' as symbols for their neutral elements. Suppose there is defined on G a function ϕ with values in H: $\phi(x) \in H$ for $x \in G$, such that

(18) $$\phi(a \top b) = \phi(a)\ \phi(b).$$

Then ϕ is called a *homomorphism* of G into H. It is easy to see that because of (18), we must have $\phi(e) = 1$ and $\phi(a^{-1}) = \phi(a)^{-1}$. However, we did not say that for $a \neq b$ we should have $\phi(a) \neq \phi(b)$. In fact, if you define $\phi(x) = 1$ for every $x \in G$, you surely have a homomorphism, but a trivial one.

The class ${}_\phi G$ of elements for which $\phi(x) = 1 \in H$ is remarkable. (It is called the kernel of ϕ.) Let $n \in {}_\phi G$ and let a be any element of G. Then $\phi(a^{-1}na) = \phi(a^{-1}) \cdot \phi(n) \cdot \phi(a) = \phi(a^{-1}) \cdot 1 \cdot \phi(a) = 1$, so that $a^{-1}na \in {}_\phi G$. Moreover, ${}_\phi G$ is a subgroup of G, for if a, $b \in G$ then $\phi(a \mathbin{\tau} b) = \phi(a) \cdot \phi(b) = 1 \cdot 1 = 1$ whence $a \mathbin{\tau} b \in {}_\phi G$; and if $a \in {}_\phi G$ then $a^{-1} \in {}_\phi G$. These properties are made the topic of a definition:

(19) A subgroup N of G is called *normal* if it is a subgroup and if $a \in G$ and $n \in N$ implies that $a^{-1} \mathbin{\tau} n \mathbin{\tau} a$ belong to N.

The second condition may be abbreviated as $a^{-1}Na = N$, or even as $Na = aN$. This doesn't mean that $n \mathbin{\tau} a = a \mathbin{\tau} n$ for every n.

We have just seen how a homomorphism ϕ gives rise to a normal subgroup ${}_\phi G$ (its kernel). We now make considerations which will lead to the theorem that for every normal subgroup N there is a homomorphism ϕ such that $N = {}_\phi G$.

We begin with an example, letting G be the abelian group of integers (writing the operation as '+') and N be the subgroup of numbers including 0 which are multiples of 6. It is normal. In fact in an abelian group, all subgroups are normal. Let g belong to G and consider the ways of writing

$$g = n + r \qquad (n \text{ in } N).$$

For example, if $g = 15$ then

$$15 = 0 + 15 = 6 + 9 = 12 + 3 = 18 - 3 = \ldots = -6 + 21 = \ldots .$$

We are interested in the residue class (the class of remainders r) that we obtain. In this example it is $\{\ldots, 21, 15, 9, 3, -3, \ldots\}$. Note that 15 occurs in the residue class it determines, and that the difference d of two elements in this residue class is always divisible by 6, *i.e.*, $d \in N$.

Now consider again a general group G. But let us omit the sign '+'. Let N be a subgroup of G. For the moment, do not require N to be normal.

Select g in G and now consider all the elements of G expressible as gn where n is in N. This class of elements is called a *residue class modulo N*, and is denoted by gN. Note that g belongs to gN, hence G is covered by residue classes. Furthermore, two residue distinct residue classes have no elements in common, for if say z belongs to both gN and hN, then $z = gn_1 = hn_2$ or $gN = zN = hN$. Of course one may have $gN = hN$ without having $g = h$.

Let the residue classes themselves be taken as the elements of a *new class*, denoted by G/N. *If N is now a normal subgroup, G/N can be made into a group to which G is homomorphic with kernel precisely N.*

Let A and B be residue classes, that is, elements of G/N. Suppose $A = gN$, $B = hN$. Then define $AB = C$ where $C = ghN$. The question is, does C depend on the choice of g and h? It does not, for if $A = g'N$, $B = h'N$, then $g' = gn_1$, $h' = hn_2$ and $g'h' = gn_1hn_2$ whence $g'h' = gn_1hN = gh(h^{-1}n_1h)N = ghN = C$. We have now defined the binary operation in G/N. We leave to the reader the verification of (13), (14), (15). In connection with (14), N itself is the neutral element of G/N, while in connection with (15), one observes that $(gN)^{-1} = g^{-1}N$. Thus G/N is a group.

The homomorphism of G onto G/N which we promised to exhibit shall be denoted by c_N, and it is given by

$$c_N(g) = gN.$$

Because of the importance of the ideas involved, we present the construction of G/N in another way. Let us write $a \sim b$ when ab^{-1} belongs to N. Then the relation '$\sim$' satisfies

(19) $$a \sim a$$

(20) $$a \sim b \text{ and } b \sim c \text{ implies } c \sim a.$$

A relation satisfying (19) and (20) is called a *congruence* relation. Note that (19) and (20) imply that $a \sim b$ when $b \sim a$. The class G on which it is defined decomposes into *congruence* classes such that two elements are $\sim$-related if and only if they belong to the same class.

The class of congruence classes is denoted by G/N. If N is a normal subgroup, then we have (21) $a \sim b$ and $c \sim d$ implies $ac \sim bd$. A relation defined in a group satisfying (21) is called a *normal* congruence relation. We leave the verification of (19), (20), (21) to the reader. It follows from these that if you take two congruence classes A and B and form all products ab, a from A, b from B, you will exactly fill some congruence class. Hence the congruence classes form a group G/N.

We turn now to another concept which we wish to describe in this article.

Next after groups, the simplest algebraic objects are probably *fields*. A *field* is a set F of at least two elements in which there are defined two binary operations + and $\cdot$ with these properties:

(22) F is an abelian group with + (denote the neutral element by 0, calling it "zero").

(23) If 0 is removed from F, the remaining elements also form a group F_0 with the operation '$\cdot$'. These relations are furthermore related by

(24) (Distributive law) for a, b, c in F

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(b + c) \cdot a = b \cdot a + c \cdot a$$

In (23) we did not mean to imply that $a \cdot 0$ was not defined, for we intend the "product" to be defined for all pairs of "factors." In fact, it follows from our axioms (22), (23), (24) that $a \cdot 0 = 0 \cdot a = 0$ for every $a \epsilon F$. To see this, denote by '1' the neutral element of the multiplicative group F_0 obtained when 0 is removed and let b be any element of F_0, perhaps 1. Then $a \cdot b = a \cdot (b + 0) = a \cdot b + a \cdot 0$ whence $a \cdot 0 = 0$. Likewise $0 \cdot a = 0$.

For a in F_0 (that is, for $a = 0$ in F), the inverse element a^{-1} is usually denoted by $\frac{1}{a}$ or $1/a$, and $a^{-1}b (= ba^{-1})$ is denoted by b/a or $\frac{b}{a}$.

The axioms of a field do not say that there is an element $z = 1/0$ such that $z \cdot 0 = 1$, for 0 does not belong to F_0. In fact, as in the case of the real or rational numbers (which are the most familiar examples of fields) there cannot exist such an element z, because as we have just shown, $z \cdot 0 = 0$. Of course $1 \neq 0$ because 1 belongs to F_0.

The axioms say nothing about there being more than the two elements 0 and 1, and there exists a field called $GF(2)$ which *has* only two elements. We exhibit the + and $\cdot$ tables:

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

The reader is urged to verify (22), (23), (24). The only strange feature of this field $GF(2)$ is that $1 + 1 = 0$ in this field. Let us exhibit a larger, but still finite field. It has 5 elements, and is called $GF(5)$. Let us denote the elements by 0, 1, 2′, 3′, 4′. In giving the + and $\cdot$ tables, we omit the rows or columns headed by 0 in the + table, and those headed by 0 or 1 in the $\cdot$ table:

+	1	2′	3′	4′
1	2′	3′	4′	0
2′	3′	4′	0	1
3′	4′	0	1	2′
4′	0	1	2′	3′

$\cdot$	2′	3′	4′
2′	4′	1	3′
3′	1	4′	2′
4′	3′	2′	1

Before we say another word, let us agree to abbreviate sums like $a + a + \ldots + a$ (m terms, m a natural number) by $m \cdot a$. This makes sense in any group in which + is the operation – hence in any field. To prevent confusion we attached primes to 2, 3, 4 in $GF(5)$. However, $3' \cdot 4' = 3 \cdot 4'$. In that sense our symbols were well-chosen. Clearly

$(m \cdot a) \cdot (n \cdot b) = mn \cdot (a \cdot b)$. Also, let $a^m = a \cdot a \cdot \ldots \cdot a$.

In the example $GF(5)$ observe that $1 + 1 + 1 + 1 + 1 = 5 \cdot 1 = 0$, also $5 \cdot 2' = 0$, etc. In fact, if $m \cdot 1 = 0$ then $(m \cdot 1)a = m \cdot a = 0 \cdot$ for every a $GF(5)$. And if $m \cdot a = 0$ where $a \neq 0$, then $a^{-1}(m \cdot a) = m \cdot 1 = 0$.

One might refuse to deal with fields in which $m \cdot 1 = 0$ for some whole number m. But this would exclude many desirable fields. In fact, if F is a finite field, then there is a natural number m such that $m \cdot 1 = 0$. To see this, begin by writing down $1, 2 \cdot 1, 3 \cdot 1, \ldots, f \cdot 1$ where f is the number of elements in F. This gives us f elements all contained in F. Hence one must be 0 or some must be equal. But if say $t \cdot 1 = s \cdot 1$ then $(t - s) \cdot 1 = 0$. In either case we see that there is a least number m such that $m \cdot 1 = 0$ in F. This is called the *characteristic* of F. In our previous example, m was equal to f, but this need not be so, as our next example shows.

Suppose without being given a multiplication or addition table we are merely told that in a given field *every* element satisfies the equation $x^4 - x = 0$. This factors into $(x - 0)(x - 1)(x^2 + x + 1) = 0$. So the field has 0 and 1 and two more elements (u, v) each of which satisfies $x^2 + x + 1 = 0$. If $1 + 1 \neq 0$ then $1 + 1 = v$ or u and so it will have to satisfy this equation: $0 = (1+1)^2 + (1+1) + 1 = (1+1+1+1) + (1+1) + 1 = 7 \cdot 1$. Thus the characteristic would be 7. But $7 > f = 4$, Hence we must conclude that $1 + 1 = 0$ and that the characteristic is 2. The sum of the roots of $x^2 + x + 1$ is $u + v = -1 = +1$ and their product is 1. We can thus fill in the tables to the following extent:

+	1	u	v
1	0	?	??
u	?	0	1
v	??	1	0

	u	v
u	???	1
v	1	!

Now $1 + u \neq 0$ because if $1 + u = 0$ then $u = 1$, and then $1^2 + 1 + 1 = 0$ whence $1 = 0$. More clearly $1 + u \neq 1, \neq u$. Hence $1 + u = v$ and we can remove '?'. Similarly '??' must be replaced by 'u'. For the other table, observe that $u^2 + u + 1 = 0$ or $u^2 = u + 1, = v$. Likewise $v^2 = u$. We get

(26)

+	1	u	v
1	0	v	u
u	v	0	1
v	u	1	0

	u	v
u	v	1
v	1	u

We can only assert that if there is such a field, then (26) describes it. It must still be checked whether (26) satisifes (22), (23), and (24). It does, and the field is called $GF(4)$.

Could there be a field with characteristic 4 or perhaps 6? No, as a matter of fact, a field F (finite or not) cannot have characteristic

$m = ij$ (where i and j are not 1), for if so then $a = i \cdot 1$ and $b = j \cdot 1$ are not 0 in F, but the product $c = ab = (i \cdot 1)(j \cdot 1) = ij \cdot 1 = m \cdot 1 = 0$. Thus a and b are *divisors of* 0, which is excluded by (23). So the characteristic of any field if it finite, is a *prime* number p.

There is a relation between the number f of elements in a finite field F and its characteristic p. We may safely designate the elements $i \cdot 1$ for $0 \leq i \leq p$ by i. Now the set $P = 0, 1, 2, \ldots, p - 1$ of these integral multiples of $1 \epsilon F$ forms a *subfield* contained in F. One sees this as follows: the sum and product of elements in P are multiples of 1 and thus in P. But now, if $a \epsilon P$ and $a \neq 0$ does $a^{-1} \epsilon P$? We use the "musical chairs" principle again: $1 \cdot a, 2 \cdot a, \ldots, (p - 1) \cdot a$ are $p - 1$ elements of P, and none of them is 0, because the coefficients are all less than p and not 0. No two can be alike, for if $i \cdot a = j \cdot a$ then $(i - j) \cdot a = 0$, but this already occurs in the set if $i > j$. Hence one of them = 1, say $i \cdot a = 1$. Then $a^{-1} = i \epsilon P$. Now obtain in F as large as possible a collection of elements $a_1, \ldots, a_n$ with the property that $m_1 a_1 + \ldots + m_n a_n = 0$ with $m_1, \ldots, m_n$ P only when $m_1 = = 0, \ldots, m_n = 0$. Then by letting $j_1, \ldots, j_n$ roam over P we obtain p^n elements of F expressible in the form.

$$A = j_1 \cdot a_1 + \ldots + j_n \cdot a_n. \tag{27}$$

Now *every* a in F is expressible in this way, for if we adjoin a to $a_1, \ldots, a_n$ we presumably get a relation

$$ma + m_1 \cdot a_1 + \ldots + m_n \cdot a_n = 0$$

with some coefficients including m, not 0. Thus $ma = -m_1 \cdot a_1 - \ldots - m_n \cdot a_n$ and multiplying by m^{-1} which is in P, we get (27). *Thus f is a power p^n of the characteristic p*, which was what we wanted to show. In $GF(4)$, $4 = 2^2$, and a_1, a_2 could be chosen as 1 and u respectively. Then $v = 1 \cdot 1 + 1 \cdot u$, $0 = 0 \cdot 1 + 0 \cdot v$.

We defined $GF(4)$ as the set of roots of $x^4 = x$. This is possible even for $GF(p^n)$. For let F_0 be the group (under multiplication) of non-0 elements. It has order $p^n - 1$. Let $x \epsilon F_0$, and let X be the subgroup of F_0 consisting of the distinct powers of x: $X = \{x, x^2, \ldots, x^{k-1}, x^k = 1\}$. The congruence classes of X cover F_0 without overlapping, so that $p^n - 1$ is a multiple of k. Hence $x^{p^n - 1} = 1$ and so $x^{p^n} = x$ for every element, even 0.

There is a generalization of the notion of a field which leads to vastly richer possibilities and more baffling problems.

A *ring* R is a set of elements in which there are defined two binary operations '+' and '·' for which (22) and (24) hold, and of which '·'

satisfies (13).

Not only don't we require the non-zero elements to form a group, we don't even require $a \cdot b = b \cdot a$, or a 1 such that $1 \cdot a = a \cdot 1 = a$, but if the former holds, R is called *commutative* (more rarely *abelian*) and if the latter holds, it is called a ring with *unity element* (the word "unit" is used in another sense for rings).

One rather silly way to get a ring is to take an abelian group G, writing the operation as '+' and the neutral element as 0 and then defining $a \cdot b = 0$ for every a and b in G. Such a ring has no 1 of course, although one can go ahead and "adjoin" one. Let us henceforth omit the '.'.

A rather important type of ring called *Boolean* ring is obtained as follows. Take any set Γ of objects (these will not form the ring). the *elements* of the ring B are to be the *subsets* of Γ. For example, if Γ is the class of all *people* in the world, then B is the class of all crowds large or small. We must include Γ itself as a "crowd" as well as the "crowd" with no people, the *empty set* Λ. If a and $b \in B$, that is to say, if they are both subsets of Γ then ab is to be the set of all. γ which belong to both a and b. Thus ab is the *common part* of a and b. The reader should now verify (13), (observe that $aa = a$). Now $a + b$ is, oddly enough, a little more complicated. After all, we must be sure that (22) and (24) are fulfilled. We define $a + b$ to be the set of all which belong either to a or to b but *not* to *both*. Λ, the void set, obviously has the property that $a + \Lambda = a$, for $\gamma \epsilon (a + \Lambda)$ if and only if it belongs to a or to Λ but not both, and since it cannot belong to Λ anyway, $\gamma \epsilon (a + \Lambda)$ if and only if $\gamma \epsilon a$, *i.e.* $a + \Lambda = a$. Therefore, we may denote 'Λ' by '0'. Observe that $a + a = 0$, whence B could be said to have characteristic 2.

This ring B is important because of its use in probability and the calculus of propositions in logic.

The most familiar ring is that of the integers, where + and − have the usual meaning. Of course, the real numbers form a ring too, but the integers are of greater interest here because they do not form a field. Suppose we consider instead the class of all integral multiples of a fixed, non-zero integer m. A moments' reflection shows that this is a ring, a subring of the ring of integers (unless $|m| = 1$); and if $|m| \neq 1$ then these rings have no 1.

A very important type of ring (*ring of endomorphisms*) is obtained as follows. Let G be an abelian group. Let R be the class of homomorphisms α, β, ... of G into itself. This means that if $a \in G$ then $\alpha(a) \in G$, and $\alpha(a + b) = \alpha(a) + \alpha(b)$. This R is a ring: we define $\alpha + \beta = \gamma$ if

$$\gamma(a) = \alpha(a) + \beta(a) \qquad (a \in G)$$

and $\alpha\beta = \gamma$ if $\gamma(a) = \alpha(\beta(a))$.

In other words $\gamma = \alpha \circ \beta$ in the very sense of this notation '$\circ$' as we used it earlier. This ring R has a 0, namely the homomorphism 0 for which $0(a) = 0 \in G$; and a unity element 1 for which $1(a) = a \in G$.

The *matrix* rings are subrings of the ring R of endomorphisms you get by taking G = euclidean n - dimensional space. In the interest of brevity we shall pass up any discussion of matrix rings, since they are explained in so many places. (That is not to deny that they are nearly the most important of all rings.)

There are many general remarks that could be made about rings, but in this paper we shall limit our discussion of rings to the consideration of homomorphisms ϕ of rings, and of the concepts to which these lead.

A *homomorphism* ϕ of one ring R to another S is a function whose domain is R and whose values are in S: $\phi(x) \in S$ for $x \in R$ with the properties

(28) $$\phi(a + b) = \phi(a) + \phi(b)$$

(note that the first '+' sign is an operation in R but the second is in S; the same thing occurred in (8), of which (28) is a special case), and, naturally enough

(29) $$\phi(ab) = \phi(a)\phi(b).$$

As in the case of groups, we are led to the kernel ${}_\phi R$ of ϕ which consists as always, of those $x \in R$ for which $\phi(x) = 0 \in S$. In the case of groups the kernel was at least a subgroup, here it is at least a subring. But as in the case of groups, an added condition crept in: the kernel was a *normal* subgroup. What condition, beyond being a subring, must the kernel of a homomorphism of a ring satisfy? Well, if $\phi(x) = 0$ then $\phi(axb) = \phi(a)\phi(x)\phi(b) = 0$ no matter how a and b are selected, that is to say, we don't need to suppose $\phi(a)$ and/or $\phi(b) = 0$. Hence the *kernel* ${}_\phi R$ satisfies the requirements of the following:

(30) a *subring* N of a ring R is *normal* if for every $n \in N$ and arbitrary a, $b \in R$, both an and na lie in N.(§)

In the case of groups we established the fact that the notions of "kernel of a homomorphism" and "normal subgroup" were co-extensive. Now (30) would not be satisfactory unless we could show the same thing here. This we can and shall do. Unless the reader has grasped the idea of *residue class* as we used it for groups, he cannot expect to follow the present considerations. Perhaps the best thing for him to do is to review now the paragraphs containing (19), (20), and (21), perhaps re-writing them using the additive notation '+' instead of the multiplicative, and supposing the group to be abelian (note how (21) becomes redundant).

* * *

(§) Warning: this notion is known as a "two-sided ideal" in the textbooks. The advanced student may appreciate our use of the term "normal subring" in this article. We are *not* trying to establish the term "normal subring" outside of this article.

Let R be a ring and N a subring. Ignoring the *multiplication* in R and N for a moment, we see that N is a subgroup (in fact a normal subgroup since R is abelian (22)). Hence we can form the class R/N of residue classes $a + N$ $(a \in R)$. These residue classes form an abelian group whose operation we may again denote by '+'.

Now let us return our attention to the ring-structure of R. We want to see if we can make R/N, which is already a group with '+' into a *ring* by defining a multiplication inherited from R. Let $a + N$ and $b + N$ be elements of R/N. Let us form the class of *all* products $a_1 b_1$ where $a_1 \in (a + N)$, $b_1 \in (b + N)$. Let us denote it by $(a + N)(b + N)$. Since a_1 is of the form $a + n_1$, and b_1 is of the form $b + n_2$, $(a + N)(b + N)$ consists of all elements of the form $ab + an_2 + n_1 b + n_1 n_2$. *Some* of these lie in the residue class $ab + N$, as you can see by choosing $n_1 = n_2 = 0$. Now the question is, do they *all* lie in $ab + N$? *If they do* then (taking n_1 and $n_2 = 0$ in turn) both $ab + n_1 b$ and $ab + an_2$ must lie in $ab + N$, *i.e.*, nb and an must lie in N, *i.e.*, (30) is necessary. Conversely, if (30) is true, that is if N is normal, then all $a_1 b_1$ *will* lie in the same residue class $ab + N$. *So the class R/N of residue classes $a + N$ forms a ring if and only if N is a normal subring.*

The zero of R/N is of course $N = 0 + N$. That is a matter of group theory. If R has a 1, then $1 + N$ is a 1 for R/N; but R/N may have a unity element even if R has none.

Now there is a group-homomorphism c_N of R onto R/N whose kernel is N and which we called the canonical homomorphism. When N is a *normal subring*, this very homomorphism preserves the multiplication as well (the reader may verify this) hence it is a homomorphism of the *ring* R whose kernel is the original normal subring N with which we started.

The reader is invited to try to find the condition N must satisfy in order that R/N be a *field*, or that R/N have no *divisors of zero*.

For further reading on the topic of this article, we point out that fortunately there exist books on this topic even in English. There is the text-book *A survey of modern algebra*, by G. Birkhoff and S. MacLane, Macmillan, 1941. Most important, this book contains exercises. No one should expect to be able to understand a subject before having put in some hard work solving problems.

University of California at Los Angeles

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known text-books should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

98. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Find a perfect cube and a perfect square such that their difference is 2,000,000.

99. *Proposed by T. E. Sydnor, Pasadena City College, California.*

a) Construct a right triangle given the inradius and the circumradius. *b*) What conditions must be placed on r and R in order that the triangle shall have integer sides?

100. *Proposed by Wang Shik Ming, Chung Hwa High School, Malang, Java, Indonesia.*

The area of the parallelogram formed by the tangents to an ellipse at the extremities of any pair of conjugate diameters is equal to the area of the rectangle contained by the axes of the ellipse.

101. *Proposed by N. A. Court, University of Oklahoma.*

If E, F are two isotomic points on the edge BC (*i.e.* equidistant from the midpoint U of BC) of the tetrahedron $DABC$, and the lines AE, DE, AF, DF meet the circumsphere again in the points P, Q, R, S, we have, both in magnitude and in sign,

$$(AE)(AP) + (DE)(DQ) + (AF)(AR) + (DF)(DS) = (BC)^2 + (AD)^2 + 4(UV)^2,$$

where V is the midpoint of DA.

102. *Proposed by Leo Moser, Texas Technological College.*

What is the least number of plane cuts required to cut a block

$a \times b \times c$ into abc unit cubes, if piling is permitted? [Suggested by **Q 12, 24,** 53, (Sept. 1950).]

103. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

A high-school student solved the linear differential equation $dy/dx + Py = Q$ for y as if it were an ordinary algebraic equation. Under what conditions could this procedure have yielded a correct solution of the differential equation?

104. *Proposed by F. L. Miksa, Aurora, Illinois.*

Given the following two arrays of the 21 combinations of seven digits taken two at a time:

Array No. 1

1	2	3	4	5	6	7
25	16	12	17	14	15	13
34	35	46	23	27	24	26
67	47	57	56	36	37	45

Array No. 2

1	2	3	4	5	6	7
23	15	12	13	14	17	16
46	36	45	27	26	24	25
57	47	67	56	37	35	34

Show that no group transformation can be found which will transform Array No. 1 into Array No. 2.

SOLUTIONS

Late Solutions

70, 71. *O. H. Hoke, University of Georgia.*

Closing Time of a Door.

69. [May 1950] *Proposed by J. S. Miller, Dillard University.*

An automobile door stands open at right angles. The hinges are at the front of the door. If the automobile is given an acceleration a find the time in which the door closes.

Solution by R. B. Leighton, California Institute of Technology and R. E. Winger, Los Angeles City College. From the viewpoint of relative motion, when the automobile starts forward with acceleration a the motion of the door, as observed by the driver, will be the same as if the hinges (assumed in a vertical line) were at rest but the door subjected to a constant force $f = ma$ toward the rear, m being the mass of the door. Since the door is constrained by the hinges its motion is analogous to that of a physical (gravity) pendulum swinging with an angular amplitude of $90°$. This is a classical problem which has been treated in many works; the general solution contains an elliptic integral.

For infinitesimal oscillations the period of a gravity pendulum is

$$T = 2\pi k/\sqrt{gd},$$

where k is the radius of gyration of the pendulum, g the acceleration due to gravity, and d the distance from the axis of rotation to the center of mass. To adapt this expression to the door we replace g by a, and note that the closing of the door is just one-quarter of a cycle of motion, hence the time is $T/4$. If the door is uniform, rectangular in shape, and of width W, $d = W/2$ and $k = W/\sqrt{3}$. This gives the required time as $T/4 = (\pi/2)\sqrt{2W/3a}$ for infinitesimal amplitude.

For finite angular amplitude α the time is

$$\frac{T}{4} = \sqrt{\frac{2W}{3a}}\left[\frac{\pi}{2}\left(1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + \frac{9}{64}\sin^4\frac{\alpha}{2} + \frac{25}{256}\sin^6\frac{\alpha}{2} + \cdots\right)\right].$$

The value of the expression in brackets is found in tables of elliptic integrals for various values of α. For $\alpha = 90^\circ$ the value is 1.8541. Hence the time required for the door to close is

$$t = T/4 = 1.8541\sqrt{2W/3a}$$

General Solution for the Period of a Physical Pendulum, by R. E. Winger. The series expression given above for the period may be developed in the following manner. The physical pendulum is a rigid body of mass m, suspended so that it can swing freely without friction about a horizontal axis which does not pass through the center of mass. If the body is given an angular displacement $\theta < \pi$ and then released it will execute periodic oscillations under the action of gravity. It is required to find the period of these vibrations.

The moment of inertia I about the axis of suspension may be written as $I = mk^2$, which defines the radius of gyration k. Let d be the distance from the axis to the center of mass. When the body is turned through an angle θ from the equilibrium position its weight will produce a restoring torque $-mgd \sin\theta$ about the axis. (The minus sign means that the torque is in a direction opposite to the displacement.) By the laws of mechanics (Newton's Second Law) this torque will produce an angular acceleration $d\omega/dt$ subject to the relation

$$I(d\omega/dt) \equiv mk^2(d\omega/dt) = -mgd \sin\theta.$$

so that

$$d\omega/dt = -(gd/k^2)\sin\theta.$$

Let $\omega = d\theta/dt$, the angular velocity. Then $d\omega/dt = (d\omega/d\theta)(d\theta/dt) = \omega(d\omega/d\theta)$. From the above relations is obtained the differential equation of motion of the pendulum:

$$\omega(d\omega/d\theta) = -(gd/k^2)\sin\theta$$

It will be convenient to express sin θ in terms of the half-angle $\theta/2$, thus

$$\omega\, d\omega = -\frac{gd}{k^2}\cdot 4 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\left[\frac{\theta}{2}\right]$$

Integration gives

$$\omega^2 = -\frac{4gd}{k^2}\sin^2\frac{\theta}{2} + C$$

To determine the constant C let the angular amplitude be α. When $\theta = \alpha$, $\omega = 0$. From these conditions

$$C = \frac{4gd}{k^2}\sin^2\frac{\alpha}{2}$$

Using this value and solving for ω we obtain

$$\omega = \frac{d\theta}{dt} = \frac{2\sqrt{gd}}{k}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}$$

On separating the variables and integrating we get

$$t = \frac{k}{2\sqrt{gd}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}$$

in which $t = T/4$, one-fourth the period of the pendulum. To put the integral into standard form it will be convenient to define a new variable ϕ by the relation

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi$$

This can be used to eliminate θ and to change the limits of integration; thus, when $\phi = 0$, $\theta = 0$, and when $\phi = \pi/2$, $\theta = \alpha$. The transformed expression becomes

$$t = \frac{k}{2\sqrt{gd}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}}$$

The integral is now in the type form of the elliptic integral of the first kind. To evaluate the integral the integrand is expanded into an infinite series by the binomial theorem, which gives

$$(1 - \sin^2\frac{\alpha}{2}\sin^2\phi)^{-1/2}d\phi = (1 + \tfrac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi + \frac{1\cdot 3}{2\cdot 4}\sin^4\frac{\alpha}{2}\sin^4\phi +$$

$$+ \frac{1\cdot3\cdot5}{2\cdot4\cdot6} \sin^6 \frac{\alpha}{2} \sin^6\phi + \cdots)d\phi.$$

The integration can be carried out simply for each term by use of the formula

$$\int_0^{\pi/2} \sin^n\phi \, d\phi = \frac{1\cdot3\cdot5\cdot7\cdots(n-1)}{2\cdot4\cdot6\cdot8\cdots n} \cdot \frac{\pi}{2}$$

if n is an even integer. Integration of the terms of the series finally gives

$$T = 4t = \frac{2k}{\sqrt{gd}} \{\frac{\pi}{2}[1 + (½)^2 \sin^2 \frac{\alpha}{2} + \left(\frac{1\cdot3}{2\cdot4}\right)^2 \sin^4 \frac{\alpha}{2} \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 \sin^6 \frac{\alpha}{2} + \cdots]\}$$

if $\sin^2 \frac{\alpha}{2} < 1$.

A Circular Locus in the Gauss Plane

76. [Sept. 1950] *Proposed by Howard Eves, Oregon State College.*

Let A, B, C be three fixed points in the Gauss plane. What is the locus of a variable point Z if (A,B,C,Z) is pure imaginary?

I. *Solution by Fred Marer, Los Angeles City College.* Without loss of generality, let the points A, B, C, Z be 0, $a + bi$, $c + di$, and $x + yi$, respectively. Then the cross-ratio

$$(A,B,C,Z) = \frac{c + di}{(c - a) + (d - b)i} \cdot \frac{(x - a) + (y - b)i}{x + yi} = K + Li$$

where

$$K = (x^2 + y^2 - ax - by)(c^2 - ca + d^2 - bd) + (bc - ad)(ay - bx).$$

Now the hypothesis that (A,B,C,Z) be pure imaginary requires that $K = 0$. Obviously then for $Z = x + yi$ to satisfy the conditions of the problem it must be on the circle $K = 0$ in the Gauss plane. If $c^2 - ca + d - bd = 0$, that is, if ACB is a right angle, the circle degenerates into a straight line.

II. *Solution by the Proposer.* Let lower case letters represent the complex coordinates of points denoted by the corresponding upper case letters. Since, by hypothesis, $(a,b,c,z) = ki$, k a real parameter, it follows that $(a,b,c,z) = -(\bar{a},\bar{b},\bar{c},\bar{z})$, or, setting $(a-c)/(b-c) = m$,

$$z\bar{z}(m + \bar{m}) - z(m\bar{a} + \bar{m}\bar{b}) - \bar{z}(\bar{m}a + mb) + (mb\bar{a} + \bar{m}a\bar{b}) = 0.$$

But this is the equation of a circle through points A and B. If we take a, b, c as turns, it is easily shown that

$$mb\bar{a} + \bar{m}a\bar{b} = m + \bar{m}.$$

It follows that the circle is orthogonal to the circumcircle of triangle ABC.

Thus the required locus is the circle through A and B and orthogonal to the circumcircle of ABC. From elementary geometry it is known that the center of this circle is at the exsymmedian point of triangle ABC opposite vertex C.

Fencing for Rectangular Pens

77. [Nov. 1950] *Proposed by H. E. Bowie, American International College.*

A farmer has a given length of fencing. He wishes to construct a given number of rectangular pens by running two fences from East to West and the necessary number of fences from North to South. If the area of the pens is to be a maximum, what relation exists between the total length of the North-South fences and the total length of the East-West fences?

Solution by L. A. Ringenberg, Eastern Illinois State College. Let $2f$ and $n - 1$ denote the length of fencing and the number of pens, respectively. Let $2x$ and ny be the total lengths of the East-West fences and of the North-South fences, respectively. Then $2f = 2x + ny$. The total area enclosed is $A = xy = (f - ny/2)y$, so $dA/dy = f - ny$. It follows that A is a maximum when $ny = f$, that is, when the total length of the North-South fences equals the total length of the East-West fences.

Also solved by *R. P. Banaugh, El Cerrito, Calif.; B. K. Gold, Los Angeles City College; R. B. Herrera, Los Angeles City College; P. N. Nagara, College of Agriculture, Thailand; A. Sisk, Maryville, Tenn.; and the proposer.*

A Special Sum of Two Triangular Numbers

78. [Nov. 1950] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

$t_r = r(r + 1)/2$ is a triangular number of order r. Find a general solution of the equation

$$t_x + t_y = z^2 + (y - z)^2.$$

Solution by P. N. Nagara, College of Agriculture, Thailand. The equation $x(x + 1)/2 + y(y + 1)/2 = z^2 + (y - z)^2$ simplifies to $x^2 + x + y = (2z - y)^2$. Let $2z - y = x + a$, where x and a are arbitrary integers.

Then $x^2 + x + y = x^2 + 2ax + a^2$, so $y = (2a - 1)x + a^2$ and $z = ax + a(a + 1)/2$, which is an integer since $a(a+1)$ is even.

Also solved by *L. A. Ringenberg, Eastern Illinois State College; and the proposer.*

A Trigonometric Equation

79. [Nov. 1950] *Proposed by Norman Anning, University of Michigan.*

Show, preferably by finding factors, that $A + B = 60°$ is a sufficient but not a necessary condition in order that we may have $\cos^2 A + \cos^2 B - \cos A \cos B = 0.75$.

Solution by R. P. Banaugh, El Cerrito, California. It is well-known that

$$\cos^2 A + \cos^2 B = \cos^2 A - \sin^2 B + 1 = \cos(A + B)\cos(A - B) + 1$$

and

$$2\cos A \cos B = \cos(A + B) + \cos(A - B).$$

When these identities are applied to $\cos^2 A + \cos^2 B - \cos A \cos B = 3/4$, we have

$$4\cos(A + B)\cos(A - B) - 2\cos(A + B) - 2\cos(A - B) + 1 = 0.$$

$$[2\cos(A + B) - 1][2\cos(A - B) - 1] = 0$$

Thus $\cos(A + B) = ½$ and $\cos(A - B) = ½$. Therefore $A + B = 60°$ is a sufficient condition, but not a necessary one since we may have also $A + B = \pi(2k \pm 1/3)$ or $A - B = \pi(2k \pm 1/3)$, $k = 0,1,2, \cdots$.

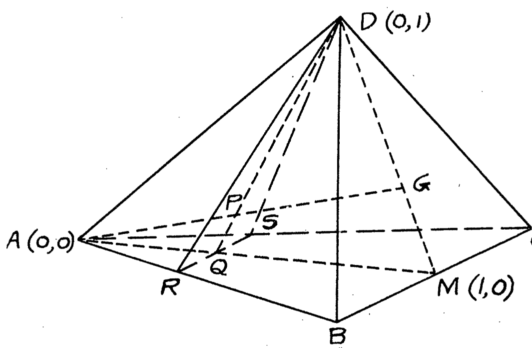
Also solved by *A. L. Epstein, Waltham, Mass.; R. E. Horton, Los Angeles City College; P. N. Nagara, College of Agriculture, Thailand; and L. A. Ringenberg, Eastern Illinois State College.*

Sectioning of Median of Tetrahedron in Rational Ratio

80. [Nov. 1950] *Proposed by R. E. Horton, Los Angeles City College.*

In tetrahedron $ABCD$ a plane is passed through vertex D and parallel to edge BC forming another tetrahedron $ARSD$. If the ratio of the volumes of these tetrahedra is a rational perfect square, then the plane cuts the median of $ABCD$ from A into two parts whose ratio is a rational number; and conversely.

Solution by Dewey Duncan, East Los Angeles Junior College. Let M be the midpoint of BC and G the centroid of BCD. Let the plane DRS meet AG in P and AM in Q. Consider AM and AD to be the x-axis and



y-axis, respectively, of an oblique system of Cartesian rectilinear coordinates with origin at 0 and with $M(1, 0)$ and $D(0, 1)$. Then G has coordinates $(2/3, 1/3)$.

Now Volume $ARSD$/Volume $ABCD$ = Area ARS/Area ABC = $(AQ)^2/(AM)^2$. So if Volume $ARSD$/Volume $ABCD$ = m^2, (m a rational number), then $AQ = m$, since $AM = 1$.

Let $AP/AG = n$. Then the coordinates of P are $(2n/3, n/3)$. Now the line DP:

$$\begin{vmatrix} x & y & 1 \\ 2n/3 & n/3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

passes through Q, which therefore, has the coordinates $[2n/(3 - n), 0]$. That is, $AQ = 2n/(3 - n) = m$. Therefore, $n = 3m/(m + 2)$ and $AP/PG = n/(1 - n) = 3m/2(1 - m)$, which is rational since m is rational.

Conversely, if $AP/PG = k$ is rational, then $m = 2k/(2k + 3)$ is rational, as is m^2 also.

Also solved by *the proposer*.

The proposition may be proven without recourse to a coordinate system. $DM/GD = 3/2$, $AQ/AM = m$ so $MQ/QA = m/(1 - m)$. Then by Menelaus' theorem, $k = AP/PG = (DM/GD)(MQ/QA) = 3m/2(1 - m)$.

The Probability of Failure

81. [Nov. 1950] *Proposed by E. P. Starke, Rutgers University.*

There is reason to suppose that in the long run and with a very large number of pupils in a properly instructed and properly tested mathematics class there would be approximately 12½% who fail to attain a satisfactory grade. Because she wants to do everything in the most modern, scientific manner, Miss X adjusts her grades in each class of 24 pupils so that there are always exactly 3 failures (and the approved number of A's, B's, etc.)

Assuming that 12½% is the correct probability of a student's failing and that Miss X teaches and tests satisfactorily, show that, on the average, less than one class (24 pupils) in four should have exactly three failures each. How often should Miss X expect a class in which half or more of the pupils ought to fail?

Solution by J. M. Howell, Los Angeles City College. The probability of failure is 12½% and the sample size is 24, so the probability of exactly 3 failures is

$$\frac{24!}{3!\ 21!}\left(\frac{7}{8}\right)^{21}\left(\frac{1}{8}\right)^{3} \doteq 0.2394.$$

Thus less than one class of 24 pupils in four should have exactly three failures.

The probability of 12 or more failing is

$$\sum_{i=12}^{24} \frac{24!}{i!(24-i)!}\left(\frac{7}{8}\right)^{i}\left(\frac{1}{8}\right)^{24-i} \doteq 0.0000091.$$

So Miss X may expect that half or more of her pupils ought to fail in less than one in one hundred thousand classes.

Also solved by *R. E. Horton, Los Angeles City College; L. A. Ringenberg, Eastern Illinois State College; and the proposer.* One incorrect solution was received.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 35. Square 85 mentally.

Q 36. Find the volume of a drinking cup which has a circular top of radius r, and whose base is a line segment of length $2r$ parallel to the top and at a distance h below it. [*Submitted by Leo Moser.*]

Q 37. Find the sum of the squares of the first n positive integers. [*Submitted by J. M. Howell.*]

Q 38. In the multiplication (746832)(592481) = 44248■770192, one digit is so badly blurred it cannot be read. What is it?

Q 39. For $x < 1$ evaluate the infinite product: $(1 + x + x^2 + \cdots + x^9)(1 + x^{10} + x^{20} + x^{30} + \cdots + x^{90})(1 + x^{100} + x^{200} + \cdots + x^{900})\cdots$. [*Submitted by Leo Moser.*]

Q 40. A square field is enclosed by a tight board fence of 11-ft. boards laid horizontally 4 boards high. The number of boards in the fence equals the number of acres in the field. What is the size of the field? [From *Civil Engineering*, **11**, 70, (January 1941).]

Q 41. Three edges of a square sheet of metal are kept at 0° each, and the fourth edge is kept at 100°. Neglecting surface radiation losses, find the temperature in the middle of the sheet. [*Submitted by Leo Moser.*]

ANSWERS

A 35. Since $(10a + 5)^2 = a(a + 1)100 + 25$, $(85)^2 = 8 \times 9 \times 100 + 25 = 7225$.

A 36. The cross-sections perpendicular to the top are triangles whose areas are clearly one-half the areas of the corresponding rectangular cross-sections of the corresponding cylinder. Hence the required volume is one-half the volume of the corresponding cylinder. Thus $V = \pi r^2 h/2$.

A 37. Consider the identity $n^3 = (n-1)^3 + 3n^2 - 3n + 1$. Then $\Sigma n^3 = \Sigma(n-1)^3 + 3\Sigma n^2 - 3\Sigma n + \Sigma 1$, so $3\Sigma n^2 = [\Sigma n^3 - \Sigma(n-1)^3] + 3\Sigma n - \Sigma 1 = n^3 + 3n(n+1)/2 - n$. Therefore, $\Sigma n^2 = (2n^3 + 3n^2 + n)/6 = n(n+1)(2n+1)/6$.

A 38. Checking by casting out 9's, we have $3 \times 2 = x + 3$, so $x = 3$.

A 39. Since every integer has a unique decimal representation, every non-negative integral power of x will appear exactly once in the product. Hence the product equals $\sum_{n=0}^{\infty} x^n = 1/(1-x)$.

A 40. Divide the square field into triangles each bounded by one chain of fence and two radial lines to the center of the square. These triangles have equal areas, so have as many boards as acres. That is, 6×4 boards and 24 acres or 240 square chains. Hence, $\frac{1}{2}\left(\frac{1}{2}s\right) \times 1 = 240$, so the side of the field is 960 chains or 12 miles.

A 41. If four such sheets are superimposed so that a $100°$ edge appears on each of the four sides, the average temperature of each side is $25°$. Hence the temperature of the middle of each sheet is $25°$. (This problem may be generalized as follows: If the edges of a regular n-gonal metal sheet are kept at temperature t_i degrees, $i = 1, 2, 3, \cdots, n$, then the temperature of the middle of the sheet is $\frac{1}{n}\sum_{i=1}^{n} t_i$ degrees.)

THE PERSONAL SIDE OF MATHEMATICS

Articles intended for this Department should be sent to the Mathematics Magazine, 14068 Van Nuys Blvd., Pacoima, California.

WHAT MATHEMATICS MEANS TO ME

T. Y. Thomas

When my friend, Professor A. D. Michal, requested that I write a short account of what mathematics means to me for publication in the Mathematics Magazine, my first reaction was to reject the proposal as utterly ridiculous. However, I finally agreed to submit a statement with the mental reservation that anything I might say on this subject would probably be considered too inane for publication.

Having decided on this course of action, I started to look into my distant past for some event which I might associate with the beginning of my interest in mathematics. The effort brought to mind a scene from my boyhood days which took place in the home of my grandfather, Dr. Titus Paul Yerkes, an elderly physician and surgeon who lived in Alton, Illinois. I do not recall the exact circumstances which lead him to speak to me about mathematics. I remember only that he explained the Pythagorean Theorem and illustrated it by drawing the squares on the hypotenuse and sides of a right angle triangle. I wish I could say that I was carried away by the beauty of this result and that I resolved henceforth to devote my life to geometry. However, my only emotion was that of complete disbelief. It was not conceivable to me at the time that anyone could be clever enough to prove a result of this character, and certainly not some Greek who lived over 2000 years ago! Still, the germ had been planted and the study of geometrical relationships later became of particular fascination.

But what about the meaning of mathematics to me? Well, we might make the rather trite observation, first of all, that mathematics is a subject studied by mathematicians, and that mathematicians are frequently considered to be divided into two classes, although these classes are not necessarily mutually exclusive. I refer to mathematics teachers and research mathematicians. At the risk of stepping ever so slightly on a few toes, I would now like to venture the opinion that of these two classes it is only the research mathematicians who are the best teachers in the fullest sense of the word.

To come more closely to grips with the subject at hand, I will say that to me mathematics is a game which must be played according to certain rules. It is also a challenge. Whenever I win the game by establishing a result or clarifying a situation in which I have come to have a deep and fundamental interest, it provides me with a sense of personal satisfaction. Usually, this sense of satisfaction or accomplishment does not last until the result in question has appeared in print, and so the game must be played again and again. Now, there are various ways in which personal satisfaction is sought and attained by humans on this planet. Some may seek it in excessive drinking, some in religious activity, others in gambling, acts of charity or philanthropy. Those who must find their satisfaction in the illusive results of mathematical research are among the most unfortunate. The rewards of mathematics are few, the standards are high, and no young man should lightly enter upon a course of study having mathematical research as its objective. He should be advised strongly to enter a firm, join the country golf club, try to break in the 90's, and finally assure his success by marrying the boss's daughter. But if he can not be diverted, if he possesses the sincerity of purpose and appears, moreover, to have the required ability, then we as professional mathematicians should encourage and aid him in every way toward his goal.

Indiana University

MISCELLANEOUS NOTES

Edited by

C. W. Robbins

Articles intended for this Department should be sent to C. W. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

SKETCHING LOCI IN POLAR CO-ORDINATES

This note discusses a method of sketching loci in polar co-ordinates. The method is not my own, nor do I know where it originated. However, the usual student reaction in a calculus class is, "Why didn't someone show us this before?" To this I have no adequate answer. I know of no text which uses the method, yet it is simple to teach and easy to use.

The method is based on the (usually well founded) assumption that the student is more familiar with the graphs of simple loci like $y = \sin 2x$ or $y = 4 + \sin 2x$ in cartesian co-ordinates than with $\rho = \sin 2\theta$ or $\rho = 4 + \sin 2\theta$ in polar co-ordinates. To sketch $\rho = \sin 2\theta$ the student sketches $y = \sin 2\theta$ on cartesian axes. The ordinates are the values of ρ for the corresponding θ in the polar sketch and may be easily obtained from the cartesian sketch.

To sketch $\rho^2 = 5 \sin 2\theta$ he first sketches $y = 5 \sin 2\theta$ in a rectangular system. Since $\rho^2 = y$, it is apparent that ρ will have two real values for each positive y and none for y negative. The desired sketches come easily and naturally, without the feeling of uncertainty on the part of the student which often arises from other methods.

The real merit of this method is the overall feeling for symmetry, periodicity, repeated values, and values of θ which lead to imaginary ρ, which the student soon acquires. The selection of appropriate limits for definite integrals is simplified. The troublesome question of why certain apparently valid selections of limits yield fallacious results is made clear without arbitrary conventions. The method also develops familiarity with "rapid sketch graphing" which is so essential in modern engineering and physics, and which is too often neglected in our teaching. Since experience speaks louder than the written word, I urge that you personally try a few graphs by this method.

University of Oklahoma — Richard V. Andree

Editor's Note: Your editor is familiar with the above method and knows others who know about it. Personally he has tried it, but finds it objectionable for several reasons. The experiences and opinions of others will be welcome.

*****

Instructor discussing logarithms with a co-ed: "You write log 314.2 = 2.4921 and then log 452 = .65514. Don't you understand that the characteristic depends upon the position of the decimal point in the number?" Co-ed: "Yes, I understand that perfectly, but I did not know what to do with log 452 because 452 does not have any decimal point."

*****

The following paragraph appears in "The Reader's Digest", Aug. 1950, P. 138.

"When you toss a coin to decide who is going to pay the check, let your companion do the calling. 'Heads' is called seven times out of ten. The simple law of averages gives the man who listens a tremendous advantage"

Henry Hoyns, quoted by Bennett Cerf in 'The Saturday Review of Literature'.

My query is - true or false?

W. A. Nichols

*****

THE SIMPLE-INTEREST RATE IMPLIED IN INSTALLMENT PAYMENTS

[*Editor's Note.* The equivalent interest rate involved in a series of installment payments may be computed by various methods, leading to different results, depending upon the assumptions made. A number of approximate formulas have been given. See, for example, E. H. Stelson, Nat. Math. Mag., 9:135-138 (Feb. 1935), and *loc. cit.*, 11:172-176 (Jan. 1937). The present paper is of interest in that it involves the use of the harmonic progression.]

On a simple interest basis, it is evident that the present value of the sum of a series of future installment payments is equal to the amount borrowed. This fact can be expressed by an equation the terms of whose left member form an harmonic progression.

Let V = the amount borrowed (the net proceeds of the loan),

I = the amount of an installment,

i = the simple interest-rate (exact rate),

F = the reciprocal of the fraction expressing the installment period in terms of one year,

N = the number of installment periods.

Then the equation is

$$\frac{I}{1+\frac{i}{F}}+\frac{I}{1+\frac{2i}{F}}+\frac{I}{1+\frac{3i}{F}}+\ldots+\frac{I}{1+\frac{(N-2)i}{F}}+\frac{I}{1+\frac{(N-1)i}{F}}+\frac{I}{1+\frac{Ni}{F}}=V.$$

Simplifying by removing the fractions, the terms of the resulting equation would contain i from the first to the Nth power. The impossibility of solving for the exact value of i in terms of the other quantities has been demonstrated by the famous Norwegian mathematician, Hendrik Abel (1802-1829). Since there is no general algebraic solution of equations higher than the fourth degree, it is the purpose of this paper to set forth a formula which closely expresses the simple-interest rate implied in a series of installment payments, the total of which exceeds the net proceeds of the loan.

Let R = the approximate rate of simple interest per annum,

A = the amount of interest (difference between the total installments paid and the net proceeds of the loan.)

The formula is:

$$R = \frac{6AF}{3IN(N+1) - 2A(2N+1)}$$

This formula, which closely approximates the simple interest-rate implied in a series of installment payments, is predicated on the following assumption, which is not wholly true: If each installment payment were discounted by a uniform rate of discount to its present value, then that rate of discount would be equivalent to a corresponding rate of interest which, when applied to each of the discounted installments, would bring the total to the amount of repayment. Of the following four illustrations, the first three test the formula by comparison of the rate of interest obtained from the solution of equations with the result yielded by the formula.

Illustration 1: X borrows \$4,628,400 and agrees to repay the loan in four equal installments of two months apart. The amount of each installment is to be \$1,214,403.84. What annual rate of simple interest is implied?

Solution: Setting up the data in the equation, we get:

$$\frac{1{,}214{,}403.84}{1+\frac{i}{6}} + \frac{1{,}214{,}403.84}{1+\frac{2i}{6}} + \frac{1{,}214{,}403.84}{1+\frac{3i}{6}} + \frac{1{,}214{,}403.84}{1+\frac{4i}{6}} = 4{,}628{,}400.$$

Solving the above equation by the trial and error method, i = 12%; by the application of the formula we get:

$$R = \frac{6 \times 229{,}215.36 \times 6}{3 \times 1{,}214{,}403.84 \times 4(4+1) - 2 \times 229{,}215.36(2 \times 4 + 1)}$$

$$= 12.0016\%$$

The difference between the two rates is 0.000016, or .0016%.

Proof of Rate of Interest of 12% Per Annum Implied:

Breakdown of *Principal Borrowed*			Amount of *Interest*	Installments *Paid*
$1,190,592.	- @12% per annum for	2 months	= $23,811.84	$1,214,403.84
1,167,696.	- @12% " " "	4 "	= 46,707.84	1,214,403.84
1,145,664.	- @12% " " "	6 "	= 68,739.84	1,214,403.84
1,124,448.	- @12% " " "	8 "	= 89,955.84	1,214,403.84
$4,628,400.			$229,215.36	$4,857,615.36

Illustration 2: *Y* borrows $4,964,350. and agrees to repay the loan in four quarterly installments of $1,332,869.44 each. What annual rate of interest is implied?

Solving by use of the equation: i = 12% (exactly);
by the use of the formula we get: R = 12.0100%.

Illustration 3: *Z* borrows $6,062,800. and agrees to repay the loan in four semi-annual installments of $1,737,111.04 each. What annual rate of simple interest is implied?

Solution by the use of the equation: i = 12% (exactly);
by the application of the formula: R = 12.0379%

Illustration 4: *G* borrows $1002.20 from the Smooth Plan Company and agrees to repay the loan in 12 monthly installments of $90. each. What annual rate of simple interest is implied?

$$R = \frac{6 \times 77.80 \times 12}{3 \times 90 \times 12(12+1) - 2 \times 77.80(2 \times 12 + 1)} = 14.65\%.$$

Emanuel Feldman
Comptroller's Office
New York City

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Contributions to Mathematical Statistics. R. A. Fisher. x + 657 pp. John Wiley & Sons, Inc. New York. 1950.

This book contains 43 research papers of R. A. Fisher which he considers his outstanding contributions to knowledge. These papers cover a wide range of subject matter that is useful to the student, statistician, mathematician, and any person who is interested in the efficient design of scientific experiment. They constitute, in the reviewer's opinion, the basis of the scientific method.

It may be of interest to the prospective reader to know that the original papers that were published in various learned journals are photographically reproduced and that the author provided a brief introductory note to place the paper in a proper perspective wherever it appeared to him necessary so as to be of assistance to the reader. Some of the original papers herein given possessed certain typographical and other errors. It appears that the reader will find that these errors have been corrected by the author and do not exist in this text. Also, certain passages in the originally published papers that to some appeared somewhat obscure have been materially clarified by the author. There is also included a brief bibliography of the author which was prepared by P. C. Mahalanobis and reprinted from *Sankhyā* (Vol. 4, Pt. 2, pp. 265-272, December 1938). In the reviewer's opinion, this bibliography should be read before reading the papers so as to obtain an invaluable picture of the life work of an outstanding international statistician. Finally, a very good subject matter index has been provided by Professor John Tukey and this is found at the end of the book.

To thoroughly appreciate the scope, content and value of this volume as well as the spread of the author's contributions to knowledge, it appears to the reviewer that the papers found in this book may be classified into three main categories: (1) contributions to the mathematical theory of statistics; (2) application of statistical theory to agriculture and the design of experiment; and (3) contributions to biology in general. More specifically, these papers discuss the foundations of theoretical statistics, the logic and theory of statistical inference, tests of significance in harmonic analysis, applications of quantitative methods in many fields of investigation, the development of various exact sampling distributions under definite assumptions, certain special designs such as the Latin and other squares, Analysis of Variance, the concept of Maximum Likelihood, uncertain inference, and Discriminant Functions.

In the reviewer's opinion, the original papers here reproduced show that Fisher was the pioneer in giving us a rigorous and unified general theory for drawing valid conclusions from statistical data. It is to be noted, however, that even today the logical and philosophical consequences are not completely worked out. Actually, statistics is the fundamental and most important part of inductive logic and mathematics, in its most advanced stages, is its tool of operation. Having this in mind, the author's contributions - the most important of which are found in this book - give a rigorous logical foundation for all inductive inferences in science where quantitative or qualitative measurements are involved. It will be found that the papers on Design of Experiment give a useful and efficient technique for getting primary data so that the maximum amount of information contained therein may be obtained in the most efficient manner. Also, the statistical papers in the field of genetics present a quantitative and mathematical foundation for biology in general.

The reviewer opines that the paper "On the Mathematical Foundations of Theoretical Statistics" is of particular interest and value. It discusses the theory appertaining to the following concepts: - (*a*) *Center of Location*. - "That abscissa of a frequency curve for which the sampling errors of optimum location are uncorrelated with those of optimum scaling"; (*b*) *Consistency*. - "A statistics is consistent, if, when it is calculated from the whole population, it is equal to the required parameter"; (*c*) *Distribution*. - "Problems of distribution are those in which it is required to calculate the distribution of one, or the simultaneous distribution of a number, of functions of quantities distributed in a known manner"; (*d*) *Efficiency*. - "The efficiency of a statistic is the ratio which its intrinsic accuracy bears to that of the most efficient statistic possible. It expresses the proportion of the total available relevant information of which that statistic makes use"; (*e*) *Efficiency Criterion*). - "The criterion of efficiency is satisfied by those statistics which, when derived from large samples, tend to a normal distribution with the smallest possible variance''; (*f*) *Estimation*. - "Problems of estimation are those in which it is required to estimate the value of one or more of the population parameters from a random sample of the population"; (*g*) *Intrinsic Accuracy*. - "The intrinsic accuracy of an error curve is the weight in large samples, divided by the number in the sample, of that statistics of location which satisfies the criterion of efficiency"; (*h*) *Isostatistical Regions.* - "If each sample be represented in a generalized space of which the observations are the coordinates, then any region throughout which any set of statistics have identical values is termed an isostatical region;" (*i*) *Likelihood.* - "The likelihood that any parameter (or set of parameters) should have any assigned value (or set of values) is proportional to the probability that if this were so, the totality of observations should be that observed" ; (*j*) *Location.* - "The location of a frequency distribution of known form and scale is the process of estimation of its position with respect to each of the several variates"; (*k*) *Optimum Value.* - "The optimum value of any parameter (or set of parameters) is that value (or set of values) of which the likelihood is greatest" ; (*l*) *Scaling.* - "The scaling of a frequency distribution of known form is the process of estimation of the magnitudes of the deviations of the several variables" ; (*m*) *Specification.* - "Problems of specification are those in which it is required to specify the mathematical form of the distribution of the hypothetical population from which a sample is to be regarded as drawn" ; (*n*) *Sufficiency.* - "A statistic satisfies the criterion of sufficiency when no other statistics which can be calculated from the same sample provides any additional information as to the value of the parameter to be estimated" ; (*o*) *Validity.* - "The region of validity of a statistics is the region comprised within its contour of zero efficiency."

In certain papers on exact sampling distributions, Fisher makes use of Euclidean hyperspace for his solution, namely, a sample of size n is represented by a point in a space of n-dimensions.

Other research of interest and importance by the author is the concept of *Discriminant Functions*. The problem here is to determine a function of the measurements that will maximize the ratio of the difference between the specific means to the standard deviations between the kinds of things.

This book is most valuable for the student in statistics as well as for the statistician, mathematician and experimentor. In the reviewer's opinion, no individual with the title *Statistician* can afford *not* to familiarize himself with such contributions as Fisher has given us. To read it, however, the individual should have considerable mathematical maturity as well as have ability to reason inductively.

The George Washington Uviversity Frank M. Weida

Statistical Decision Functions. Abraham Wald, ix plus 179 pp. $5.00. John Wiley & Sons, New York, 1950.

This book is a very valuable and important addition to the mathematical foundations of theoretical statistics. It appears that the theory of statistical decision functions is a generalization of earlier theories appertaining to the testing of hypotheses. These earlier theories seem to be restricted by the fact that experimentation to a large extent was considered to be processed in a single stage and the actual decision problems involved testing a hypothesis and point and interval estimation. The general theory developed in this book does not possess these restrictions.

It is well known, and the author clearly states, that any decision problem involves a stochastic process. Such a process is a collection of chance variables which have a joint probability distribution. In any experimentation, the necessity arises to make a choice from a set of alternative decisions. Each decision, of course, has a degree of preference as well as risk which depends on the unknown distribution $F(X)$ after a certain amount of experimentation.

Wald states the decision problem very clearly as follows: "Given (1) the stochastic process $\{X_i\}$, (2) the class Ω of distributions which is known to contain the true distribution $F(X)$ as an element, (3) the space D^t of possible terminal decisions, (4) the weight function $W(F, d^t)$ defined for all elements F of Ω and all elements d^t of D^t, and (5) the cost function $c(x; s_1, \ldots, s_k)$ of experimentation, the problem is to choose a decision function $\delta(x; s_1, \ldots, s_k)$ to be adopted for carrying out the experiment and for making a terminal decision." To determine the merit of a particular decision function, Wald introduces the important risk function for the particular decision function.

The book is divided into five chapters. In Chapter I, the author treats the general decision problem and various of its basic concepts. The consequences of the adoption of a particular decision function including its performance characteristic, its risk and cost functions are clearly and rigorously developed. The author also shows that certain earlier methods of testing a hypothesis and point and interval estimation are special cases of the general decision problem. Another interesting point is that the decision problem is viewed as a Zero Sum Two-Person game in the sense of von Neumann. Chapter II gives a generalization of von Neumann's theory of Zero Sum Two-Person games. In Chapter III there is developed a general theory of statistical decision functions. The reader will find that the generalization presented in Chapter II is used in this general theory. Chapter IV gives additional general theory – the results of earlier research by the author and Wolfowitz. Essentially, the general theory is developed for the case where the chance variables are independently and identically distributed and the cost of experimentation is proportional to the number of observations. In Chapter V applications to various special problems are discussed. The author discusses (a) some non-sequential decision problems – problems where the probability is unity that experimentation is carried out in a single stage; (b) some specific sequential decision problems. Amongst the latter, Wald discusses a two-sample procedure for testing the mean of a normal distribution, a sequential procedure for testing the means of a pair of binomial distributions, and a decision problem when the class of distributions consists of three rectangular distributions.

Throughout the book, the author emphasizes generalized ideas and concepts. In the reviewer's opinion, it is regrettable that not more attention is given specific methods, techniques and applications. Greater emphasis on the latter phase would make the text more valuable to the applied worker.

To understand and appreciate the beautiful and most valuable and useful theory presented in this book, the reader should have knowledge of Advanced Calculus, the Theory of Aggregates and Point Sets, the Theory of Measure - particularly in the sense of Borel, Set Functions, Integration Theories, and a thorough knowledge of Mathematical Probability and the elements of Mathematical Statistics so as to appreciate, understand and recognise inductive processes and behavior.

The material in this book is presented in a logical, rigorous and clear manner. We have in this book a real contribution and the reviewer is of the opinion that every statistician should read and digest this valuable and useful contribution to knowledge. Only a scholar could do what Wald has given us.

The George Washington University Frank M. Weida

Fourier Methods. By P. Franklin. New York, McGraw-Hill Book Company, 1949. x + 289 pages. $4.00.

This book serves as a fine introduction to Fourier series and Laplace transforms, based only on a first course in calculus. No attempt at mathematical rigor is made, as the primary purpose here is to enable the reader to acquire some manipulative skill in Fourier methods, which refer to "any analysis or synthesis of functions by a linear process applied to sines, cosines, or to complex exponentials". At the end of each chapter there is an adequate list of references which the interested reader can utilize in pursuing theoretical points or further physical applications.

The five chapters are: 1. Complex quantities. Impedance. 2. Fourier series and integrals. 3. Partial differdntial equations. 4. Boundary value problems. 5. Laplace transforms. Transients.

Numerous physical applications are discussed in detail throughout the work, and a wealth of problems can be found in the thirty-one sets of exercises. The reader should not fail to look at these examples, as some new material is presented there in the form of problems.

A first edition, this book appears to contain many minor and easily detectable misprints. A good feature is the treatment of Fourier series for the general interval instead of one scaled down to 2π. The treatment of Heaviside's operational calculus in Chapter 5 is quite enlightening. A slightly disturbing feature is the occurrence of two entities long before the definitions themselves appear.

It is the impression of the reviewer that this book would be quite satisfactory as a text for a one-semester course for senior or first-year graduate students in applied mathematics, engineering, and physics.

University of Arizona Louise H. Chin

OUR CONTRIBUTORS

A biographical sketch of *E. T. Bell* appeared in the January-February issue, 1951.

Robert C. James, Assistant Professor of Mathematics, University of California, was born in Indiana in 1918. He attended the University of California at Los Angeles (B. A. 1940) and California Institute of Technology (Ph. D. 1946), was Benjamin Pierce Instructor in Mathematics at Harvard 1946-1947. He has published several papers on the theory of Banach spaces, linear topological spaces, and topological groups. He is one of the editors of the "Mathematics Dictionary".

W. W. Funkenbusch, Assistant Professor of Mathematics, Michigan College of Mining and Technology, was born in Canton, Missouri in 1918. Educated at Culver-Stockton (B.A. '40) and Oregon State (M.S. '50) he taught mathematics at Oregon State College and Michigan State College before taking his present position. In addition he has had some experience as a seismographic computer for the Western Geophysical Company.

Alan Wayne was born in New York in 1909, and attended the college of the City of New York (B.S. '31; M.S. '37) and Columbia University. He has taught in the New York high schools, at the Cooper Union School of Engineering and also has held instructorship in education at New York University. At present he is First Assistant in Mathematics and Science at the Williamsburgh Vocational High School, Brooklyn, N.Y. A confessed puzzle addict, he has been President of the New York Riddlers Club and is a charter member of the N.Y. Cipher Society. Mr. Wayne has also served as President of the Association of Teachers of Mathematics of N.Y.C.

Tracy Yerkes Thomas, Professor and Chairman, Department of Mathematics, Indiana University, was born in Alton, Illinois, in 1899. After attending the Rice Institute (A.B. '21), and Princeton (M.A. '22; Ph.D. '23) he became a National Research Fellow and studied at Chicago, Zürich, Harvard, and Princeton. He joined the Princeton faculty in 1926 and was made associate professor in 1931. In 1938 he became Professor of Mathematics at U.C.L.A., and accepted his present position as chairman of the department at Indiana in 1944. Dr. Thomas is a member of the National Academy of Sciences, and served as Vice-President of the American Mathematical Society from 1940-42. He is the author of two books on tensors and differential invariants, as well as the coauthor of two elementary texts on algebra and trigonometry, and has contributed numerous research papers on differential geometry, tensor analysis, and on applied mathematics. He has recently been made head of the Graduate Institute for Applied Mathematics at Indiana University.

(Continued on P. 252)